Graphical regular representations of groups of prescribed valency

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König’s conjecture

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König conjectured in his 1936 book “Theorie der endlichen und unendlichen Graphen’, the first textbook on the field of graph theory, that every finite group is the automorphism group of a finite graph.
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- In 1957, Sabidussi\(^3\) proved that for all integers \(k \geq 3\), every finite group is the automorphism group of a \(k\)-valent graph.

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In 1949, Frucht\textsuperscript{2} proved a stronger version stating that every finite group is the automorphism group of a cubic graph.

In 1957, Sabidussi\textsuperscript{3} proved that for all integers $k \geq 3$, every finite group is the automorphism group of a $k$-valent graph.

In the above results, the group may not act transitively on the vertex set and may not have the same order as the graph.

\textsuperscript{1}R. Frucht, Herstellung von Graphen mit vorgegebener abstrakter Gruppe, \textit{Compositio Math.}, 6 (1939), 239–250.


\textsuperscript{3}G. Sabidussi, Graphs with given group and given graph-theoretical properties, \textit{Canadian J. Math.}, 9 (1957), 515–525.
A graph $\Gamma$ is called a graphic regular representation (GRR) of a group $G$ if $\text{Aut}(\Gamma) \simeq G$ acts regularly on the vertex set of $\Gamma$.

Thus a GRR of a group $G$ is a Cayley graph of $G$ with smallest possible automorphism group: $\text{Cay}(G, S)$ is a GRR of $G$ iff $\text{Aut}(\text{Cay}(G, S)) \simeq G$.

In this case, it is easy to see that $S$ is a generating set of $G$.

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After a long series of papers by various authors including Imrich, Nowitz and Watkins, the question of which finite groups have GRRs was answered by Godsil\(^4\).

A recent result of Conder and Poznanovic shows that for every integer $k \geq 3$, there exists a $k$-valent GRR of some group. However, a Sabidussi-like theorem concerning GRRs of a prescribed valency is still far out of reach— even for a Frucht-like theorem on cubic GRRs. There is special interest in cubic GRRs of finite nonabelian simple groups partially because of the related generation problem of finite nonabelian simple groups, the fact that every finite insoluble group has a GRR. In 2002, Fang, Li, Wang and Xu conjectured that every finite nonabelian simple group has a cubic GRR.
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- the fact that every finite insoluble group has a GRR.

In 2002, Fang, Li, Wang and Xu\(^6\) conjectured that every finite nonabelian simple group has a cubic GRR.


The Fang-Li-Wang-Xu conjecture

The alternating group $A_n$ with $n \geq 5$ has a cubic GRR (Godsil 1983).

The Suzuki group $2B_2(q)$ with $q = 2^{2c+1} \geq 8$ has a cubic GRR (Fang-Li-Wang-Xu 2002).

The 2-dimensional projective special linear group $PSL_2(q)$ with $q \geq 4$ has a cubic GRR iff $q \neq 7$ (Fang-X. 2016).

In particular, $PSL_2(7)$ is a counterexample to the Fang-Li-Wang-Xu conjecture.

Conjecture (Fang-X. 2016)
There are only finitely many finite nonabelian simple groups that have no cubic GRR.
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More on cubic GRRs of $\text{PSL}_2(q)$

If $\text{Cay}(G, S)$ is a cubic GRR of $G$, then $S$ either consists of three involutions or contains exactly one involution.

Theorem (Fang-X. 2016)

Let $G = \text{PSL}_2(q)$ with $q \geq 4$.

(a) If $\text{Cay}(G, S)$ is a cubic GRR of $G$, then $S$ is a set of three involutions.

(b) If $q \neq 7$, then there exist three involutions $x, y, z$ in $G$ such that $\text{Cay}(G, \{x, y, z\})$ is a cubic GRR of $G$.

(c) There exist involutions $x$ and $y$ in $G$ such that the probability for a randomly chosen involution $z$ to make $\text{Cay}(G, \{x, y, z\})$ a cubic GRR of $G$ tends to 1 as $q$ tends to infinity.
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Inspired by these results, Spiga made some conjectures:

(i) Except for a finite number of cases and for the groups $\text{PSL}_2(q)$, every finite nonabelian simple group $G$ contains an element $x$ and an involution $y$ such that $\text{Cay}(G, \{x, x^{-1}, y\})$ is a cubic GRR of $G$.

(ii) Except for a finite number of cases, every finite nonabelian simple group $G$ contains three involutions $x, y$, and $z$ such that $\text{Cay}(G, \{x, y, z\})$ is a cubic GRR of $G$.

(iii) The proportion of cubic Cayley graphs (up to isomorphism) over a finite nonabelian simple group $G$ that are GRRs tends to 1 as $|G|$ tends to infinity.
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(iii) The proportion of cubic Cayley graphs (up to isomorphism) over a finite nonabelian simple group $G$ that are GRRs tends to 1 as $|G|$ tends to infinity.
Theorem (X. 2017+)

Let $G$ be a finite simple group of Lie type of rank at least 9. Then there exists an element $x$ of prime order in $G$ such that the probability for a random involution $y$ in $G$ to make $Cay(G, \{x, x^{-1}, y\})$ a cubic GRR of $G$ tends to 1 as $|G|$ tends to infinity.

The theorem gives an affirmative answer to Spiga's conjecture (i) for finite simple groups of Lie type of rank at least 9, and also gives evidence for Spiga's conjecture (iii).

The theorem implies that there are at most finitely many finite simple groups of Lie type of rank at least 9 that have no cubic GRR, which reduces the verification of our conjecture "Only finitely many finite nonabelian simple groups have no cubic GRR" to finite simple groups of Lie type of rank at most 8.
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More remarks

As a biproduct of the theorem, for a finite simple group $G$ of Lie type of rank at least 9, there exists an element $x$ of prime order in $G$ such that the probability for a random involution $y$ in $G$ to make $\{x, y\}$ a generating set of $G$ tends to 1 as $|G|$ tends to infinity.
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This is an asymptotic version of a recent result of King\textsuperscript{7} that every finite nonabelian simple group can be generated by an involution and an element of prime order.

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- The theorem can be generalised from valency three to valency odd prime (X. 2018+).

Thank you for listening!