Flag-transitive automorphism groups of 2-designs

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A 2-$(v, k, \lambda)$ design is a pair $\mathcal{D} = (\mathcal{P}, \mathcal{B})$, such that

(a) $\mathcal{P}$ is a $v$-set, the elements is called points;
(b) $\mathcal{B}$ is a collection of $b$ $k$-subsets of $\mathcal{P}$, its element called blocks;
(c) each 2-subset of $\mathcal{P}$ is contained in exactly $\lambda$ blocks.

Furthermore, if $b = v$, then $\mathcal{D}$ is called symmetric, otherwise is non-symmetric.
Basic facts on 2-designs

The numbers $v, b, r, k, \lambda$ are parameters of the 2-design. It is well known that

$$bk = vr,$$

$$b \geq v \text{ (Fisher’s inequality),}$$

and so

$$r \geq k.$$
Automorphism groups of designs

An automorphism of a design $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ is a permutation $\pi$ of $\mathcal{P}$ such that $B \in \mathcal{B}$ implies $\pi(B) \in \mathcal{B}$.

All automorphisms of $\mathcal{D}$ form a group which acts on $\mathcal{P}$, denoted by $Aut(\mathcal{D})$.

If $G \leq Aut(\mathcal{D})$ then $G$ is a automorphism group of $\mathcal{D}$. Then $G$ is called

- point-transitive (primitive): if $G$ is transitive (primitive) on $\mathcal{P}$;
- block-transitive (primitive): if $G$ is transitive (primitive) on $\mathcal{B}$;
- flag-transitive: if $G$ is transitive on the set of flags $\mathcal{F} = \{(\alpha, B) | \alpha \in \mathcal{B}\}$;
- antiflag transitive: If $G$ acts transitively on the set of anti-flags $\{(\alpha, B) | \alpha \notin B\} \subseteq \mathcal{P} \times \mathcal{B}$;
- ...


2 — \((v, k, \lambda)\) designs

1) \(2 - (v, k, 1)\) designs: It is also called a finite linear space.
2) Lots of work have been done on flag-transitive symmetric 2-designs. Although there exists large families of non-symmetric 2-designs, less is known when \(\mathcal{D}\) is non-symmetric admitting a flag-transitive automorphism group. Here we study flag-transitive non-symmetric 2-designs. So we have

\[
b > v,
\]

and

\[
r > k.
\]
2 — \((v, k, \lambda)\) designs

3) The complement \(\overline{D}\) of a \(2-(v, k, \lambda)\) design \(D = (P, B)\) is a \(2-(v, v - k, b - 2r + \lambda)\) design \((P, \overline{B})\), where \(\overline{B} = \{P \setminus B \mid B \in B\}\).

4) It is easily known that \(G \leq Aut(D)\) is flag-transitive on \(D\) if and only if \(G\) is antiflag transitive on \(\overline{D}\).
Flag-transitive projective planes

Symmetric design with $\lambda = 1$ are projective planes. If $G \leq Aut(D)$ is flag-transitive, the classification has been done by Kantor [5] (1987, J. Alg. 106, 15-45).

Theorem

If $D$ is a projective plane of order $n$ admitting a flag-transitive automorphism group $G$, then either:

(1) $D$ is Desarguesian and $G \leq PSL(3, n)$, or
(2) $G$ is a sharply flag-transitive Frobenius group of odd order $(n^2 + n + 1)(n + 1)$ and $n^2 + n + 1$ is a prime.
Flag-transitive 2-$(v, k, 1)$ Designs

Flag-transitive 2-designs with $\lambda = 1$ are classified by BDDKLS in 1991 except 1-dimension affine case, which later done by Saxl and Liebeck.
Theorem
Let $G$ be a flag-transitive automorphism group of a linear space $\mathcal{D}$. Then one of the following holds:

1. $G \leq \AGL(1, p^a)$, for some prime power $p^a$.
2. $\mathcal{D}$ is an affine translation plane of dimension at least 2.
3. $\mathcal{D}$ is $W(q)$ and $\PSL(2, 2^a) \leq G \leq \PGL(2, 2^a)$.
4. $\mathcal{D}$ is either a projective or affine design.
5. $\mathcal{D}$ is the affine plane over the nearfield of order 9.
6. $\mathcal{D}$ is the Hering affine plane of order 27.
7. $\mathcal{D}$ is one of two designs having $v = 9^2$ and $k = 9$.
8. $\mathcal{D} = U_H(q)$ and $\PGL(q) \supseteq G \supseteq \PSU(3, q)$.
9. $\mathcal{D} = U_R(q)$ and $\Aut(\mathcal{G}_2(q)) \supseteq G \supseteq \mathcal{G}_2(q)$ where $q = 3^{2e+1}$. 
Flag-transitive 2-designs with \((r, \lambda) = 1\)

In 1988, P. H. Zieschang studied on flag-transitive 2-designs with \((r, \lambda) = 1\). He proved the following

**Theorem**

*If \(G\) is a flag-transitive automorphism group of a 2-design with \((r, \lambda) = 1\) and \(T\) is a minimal normal subgroup of \(G\), then \(T\) is abelian, or simple and \(C_G(T) = 1\).*

From this result we known that \(G\) is affine or almost simple, so we can using CFSG to consider the classification of this type of designs.
Based on this Theorem, one natural question is:

**Question**

*Can we classify the flag-transitive 2-designs with \((r, \lambda) = 1\) and \(T\) is nonabelian simple group, especially \(T = A_n\)?
$(r, \lambda) = 1$: Symmetric Case

In 2014, Zhu, Guan and Zhou classified flag-transitive SD with $(r, \lambda) = 1$ and $Soc(G) = A_n$.  

**Theorem**

(Zhu-Guan-Zhou, 2015, Front. Math. China) If $\mathcal{D}$ is a $(v, k, \lambda)$ symmetric design with $(r, \lambda) = 1$, which admits a flag-transitive automorphism group $G$ with alternating socle, then $\mathcal{D}$ is the projective space $PG_2(3, 2)$ and $G = A_7$. 
$(r, \lambda) = 1$: Non-Symmetric Case

For Non-symmetric Case, our main result is the following.

**Theorem**  
*(Zhou and Wang, 2015, EJC)* Let $\mathcal{D}$ be a *non-symmetric* $2-(v, k, \lambda)$ design with $(r, \lambda) = 1$, where $r$ is the number of blocks through a point. If $G \leq \text{Aut}(\mathcal{D})$ is flag-transitive with alternating socle, then up to isomorphism $(\mathcal{D}, G)$ is one of the following:

(i) $\mathcal{D}$ is a unique $2-(15, 3, 1)$ design and $G = A_7$ or $A_8$.

(ii) $\mathcal{D}$ is a unique $2-(6, 3, 2)$ design and $G = A_5$.

(iii) $\mathcal{D}$ is a unique $2-(10, 6, 5)$ design and $G = A_6$ or $S_6$. 
Corollary

Together with symmetric and non-symmetric cases, we have the following classification result:

**Corollary**

*If* $D$ *is a* $2-(v, k, \lambda)$ *design with* $(r, \lambda) = 1$, *which admits a flag-transitive automorphism group $G$ with alternating socle, then* $D$ *is one of the following:*

1. *a* $2-(6, 3, 2)$ *design;*
2. *a* $2-(10, 6, 5)$ *design;*
3. *the projective space* $PG(3, 2)$;
4. *the projective space* $PG_2(3, 2)$. 
$(r, \lambda) = 1$: SD, Affine Case

Recently, M. Biliotti and A. Montinaro (JCD, 2017, 25(2): 85-97) classified symmetric 2-designs with $(k, \lambda) = 1$, admitting a flag-transitive automorphism group of affine type.

**Theorem**

Let $D = (P, B)$ be a nontrivial $2 - (v, k, \lambda)$ symmetric design with $(r, \lambda) = 1$. If $D$ admits a flag-transitive automorphism group $G$ of affine type, then $v = p^d$, $p$ an odd prime, and $G$ is a point-primitive, block-primitive subgroup of $AGL_1(p^d)$. 
Furthermore, the following hold:
1. $O(G)$ acts flag-transitively and point-primitively on $\mathcal{D}$. This group has the form $T\langle \bar{\omega}^i, \sigma^y \bar{\omega}^j \rangle$, where $y \mid d$, $i \mid (p^d - 1, j \frac{p^d - 1}{py - 1})$ and $\bar{\omega}: x \rightarrow \omega x$, $\omega$ is a generator of $GF(p^d)^*$, and $\sigma: x \rightarrow x^p$.
2. There exists a divisor $x$ of $d/y$, such that $\mathcal{D}$ is the development of a $(p^d, x \frac{p^d - 1}{i}, x^2 \frac{p^d - 1 - i/x}{i^2})$-difference set. There exists $D \in \mathcal{D}$ of the form

$$D = \bigcup_{i=0}^{x-1} C_{it}$$

(disjoint union). Moreover, $D \subseteq \langle \omega^{2^a} \rangle$ and $2^a \mid i$, where $2^a$ is the maximal 2-power dividing $p^d - 1$. If $x > 1$, then $2^a \mid j$ too.
In 2013, Doc. Tian and Zhou classified the case of $\text{Soc}(G)$ is a sporadic simple group:

**Theorem**

Let $\mathcal{D}$ be a non-trivial $2-(v, k, \lambda)$ symmetric design with a flag-transitive, point-primitive automorphism group $G$ of almost simple type and with $\text{Soc}(G)$ a sporadic simple group. Then $\mathcal{D}$ and $G$ are one of the following:

(i) $\mathcal{D}$ is a $2-(144, 66, 30)$ symmetric design and $G = M_{12}$ or $M_{12} : 2$;

(ii) $\mathcal{D}$ is a $2-(176, 50, 14)$ symmetric design and $G = HS$;

(iii) $\mathcal{D}$ is a $2-(176, 126, 90)$ symmetric design and $G = M_{22}$ or $HS$;

(iv) $\mathcal{D}$ is a $2-(14080, 12636, 11340)$ symmetric design and $G = Fi_{22}$. 
But for non-symmetric case, and $Soc(G)$ be sporadic, we have been classified for

(1) $\lambda = 2, 3, 4$ or 5.

(2) $(r, \lambda) = 1$, (Zhan-Zhou);
Liang’s Results: Nonsymmetric 2-$(v, k, 2)$ Designs

Dr. Hongxue Liangobtain the following results for flag-transitive, point-primitive nonsymmetric 2-$(v, k, 2)$ designs:

(i) (with Zhou, 2016, JCD) $G$ must be affine or almost simple.

(ii) (with Zhou, 2016, JCD) If $Soc(G)$ is sporadic, then $D$ be a unique 2-$(176, 8, 2)$ designs with $G = HS$.

(iii) (with Zhou, 2016, BBMS) If $Soc(G) = A_n$, then $D$ be a 2-$(6, 3, 2)$ design ($G = A_5$) or a 2-$(10, 4, 2)$ design ($G = S_5, A_6$ or $S_6$).

(iv) (with Devillers, Praeger, Xia) If $Soc(G) = L_n(q) (n \geq 3)$, then $D$ be a 2-$(\frac{3^n-1}{2}, 3, 2)$ design with socle $PSL(n, 3)$.
Recently, Zhan and Zhou (ARS Math. Contemp., 2018) study 2-design with \( \lambda \geq (r, \lambda)^2 \), admitting a flag-transitive automorphism group.

**Theorem**  
Let \( D \) be a 2-(\( v, k, \lambda \)) design with \( \lambda \geq (r, \lambda)^2 \). If \( G \) is a flag-transitive automorphism group of \( D \), then \( G \) is of

1. **affine,**
2. **almost simple type,** or
3. **product type** with \( \text{Soc}(G) \cong T \times T \), where \( T \) is a nonabelian simple group and \( G \) has rank 3.
\[ \lambda \geq (r, \lambda)^2: \text{Symmetric design} \]

Applied this to symmetric 2-designs we get:

**Theorem**

Let \( D \) be a 2-(\( v, k, \lambda \)) symmetric design with \( \lambda \geq (r, \lambda)^2 \), which admits a flag transitive automorphism group \( G \). Then \( G \) is an affine or almost simple group.

**Question**: How about the non-symmetric case?
Recently, Huiling Li, Zhilin Zhang and S. Zhou proved the following result:

**Theorem**

Let $\mathcal{D}$ be a $2-(v, k, \lambda)$ design with $\lambda \geq (r, \lambda)^2$ which admits a flag-transitive automorphism group $G$. Then $G$ is not of product type.
Proof

Let $\Omega = \Delta_1 \times \Delta_2$, where $\Delta_i = \Delta$ and $|\Delta| = \omega$. Let $T^2 \leq G \leq H \wr S_2$, where $T = Soc(H)$ is simple, nonabelian and $H$ acts 2-transitively on $\Delta$. Let $T^2 = T_1 \times T_2$ with $T_1 \cong T_2$. Here $T_i = Soc(H_i)$ acts 2-transitively on $\Delta_i$ for $i = 1, 2$. Furthermore, $G$ is a primitive permutation group of degree $\omega^2$ and rank 3 with the suborbits

$$
\Omega_0 = \{(\alpha, \alpha)\}, \\
\Omega_1 = \{(\alpha, \beta), (\beta, \alpha) \mid \beta \in \Delta, \alpha \neq \beta\}, \\
\Omega_2 = \{(\beta, \gamma) \mid \beta, \gamma \in \Delta, \beta \neq \alpha, \gamma \neq \alpha\}
$$

of lengths $1, 2(\omega - 1), (\omega - 1)^2$. We will prove the result in several steps.
Step 1. \( r^* = 2(\omega - 1) \) where \( r^* = \frac{r}{(r,\lambda)} \).

Step 2. \( k = \frac{(c+d-2)(\omega+1)}{2} + 1 \).

Step 3. \( \lambda^* = c + d - 2 \) where \( \lambda^* = \frac{\lambda}{(r,\lambda)} \).

Step 4. \( d = c + 1 \).
Step 5. $b < 4\omega^2$.

Let

$$L_1 = \{g \in H_1 \mid (g, 1) \in G_B\},$$

$$L_2 = \{h \in H_2 \mid (1, h) \in G_B\},$$

which is the kernel of $G_B$ acting on $B^2, B^1$ respectively. Let

$$M_1 = \{g \in H_1 \mid \text{there exists } h \in H_2 \text{ such that } (g, h) \in G_B\},$$

and $M_2$ can be defined similarly. Clearly, $M_i$ is the projection of $G_B$ on $H_i$. Hence, we have the following structure of $G_B$:

Step 6. $L_1 \times L_2 \trianglelefteq G_B \trianglelefteq M_1 \times M_2$.

Step 7. $|H_i : M_i| \geq \omega$ for $i = 1, 2$. 
Step 8. $G_B = M_1 \times M_2$.

Step 9. The final contradiction.

By above analysis, we know that the structure with the point set $\Delta_1$ and the block set $\{(B^1)^g \mid g \in H_1\}$ is a 2-$(\omega, k_1, \lambda_1)$ design $D_1$, the structure with the point set $\Delta_2$ and the block set $\{(B^2)^g \mid g \in H_2\}$ is a 2-$(\omega, k_2, \lambda_2)$ design $D_2$. Let $b_i, r_i$ be the parameters of $D_i$.

Recall that $G_B = M_1 \times M_2$, $B^1$ is the orbit of $M_1$ and $B^2$ is the orbit of $M_2$. If $\alpha \in B^1$, $\beta \in B^2$, then $(\alpha, \beta) \in B$. On the other hand, for any point $\alpha' \in B^1$ and $\beta' \in B^2$, there exists $g \in M_1$ and $h \in M_2$ such that $\alpha' = \alpha^g$, $\beta' = \beta^h$. It follows that $(\alpha', \beta') \in B$, and hence $B = B^1 \times B^2$. 
Hence, the block of $\mathcal{D}$ has the form $(B^1)^g \times (B^2)^h$, where $g \in H_1$, $h \in H_2$, which is the direct product of a block (denoted by $X$) of $\mathcal{D}_1$ and a block (denoted by $Y$) of $\mathcal{D}_2$. Clearly, if $\alpha \in \Delta_1, \beta \in \Delta_2$, then $(\alpha, \beta) \in X \times Y$ if and only if $\alpha \in X$ and $\beta \in Y$. Let $\alpha, \gamma \in \Delta_1$ with $\alpha \neq \gamma$, and let $\beta, \delta \in \Delta_2$ with $\beta \neq \delta$. Then $p = (\alpha, \beta), q = (\alpha, \delta)$ and $s = (\gamma, \delta)$ are three different points of $\Omega = \Delta_1 \times \Delta_2$. Therefore, $p, q \in X \times Y$ if and only if $\alpha \in X$ and $\beta, \delta \in Y$. There are $r_1$ possibilities for $X$ and $\lambda_2$ possibilities for $Y$, so that $p, q$ are contained in $r_1 \lambda_2$ blocks of $\mathcal{D}$. On the other hand, $p, s \in X \times Y$ if and only if $\alpha, \gamma \in X$ and $\beta, \delta \in Y$. There are $\lambda_1$ possibilities for $X$ and $\lambda_2$ possibilities for $Y$, so that $p, s$ are contained in $\lambda_1 \lambda_2$ blocks of $\mathcal{D}$. However, $r_1 \lambda_2 > \lambda_1 \lambda_2$ for $r_1 > \lambda_1$, and hence $\mathcal{D}$ is not a 2-design.
Recently, Zhang Zhilin and S. Zhou (Designs Codes and Cryptography, 2017, online) studied flag-transitive, point-quasiprimitive $2-(v, k, \lambda)$ designs with $\lambda \leq 5$.

**Theorem**

Let $\mathcal{D}$ be a $2-(v, k, 2)$ design which admits a flag-transitive, point-quasiprimitive automorphism group $G$. Then $G$ is of holomorph affine or almost simple type.
A question naturally arise from this Theorem: are there any flag-transitive 2-$(v, k, 2)$ designs admitting a quasiprimitive, but imprimitive automorphism group? The answer is no, and we have the following.

**Corollary**

Let $D$ be a 2-$(v, k, 2)$ design which admits a flag-transitive automorphism group $G$. Then $G$ is primitive if and only if it is quasiprimitive.
Thank You!


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