CUBIC SYMMETRIC GRAPHS VIA RIGID CELLS

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Joint work with M. D. E. Conder, A. Hujdurović and D. Marušič

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Every component of $X[\text{Fix}(\alpha)]$ is referred to as an $\alpha$-rigid cell.
We will use the terms $I$-tree, $H$-tree, $Y$-tree, $A$-tree and $B$-tree for the graphs given in the figure. More precisely, we will denote these graphs by $I(u,v)$, $H(u,v)$, $Y(v)$, $A(v)$ and $B(v)$, respectively.
17 types of cubic symmetric graphs  
(Conder, Nedela, 2009)

<table>
<thead>
<tr>
<th>$s$</th>
<th>Type</th>
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<td>${1}$</td>
<td>Sometimes</td>
<td>3</td>
<td>${1^1, 3}$</td>
<td>Never</td>
<td>5</td>
<td>${1, 4^1, 4^2, 5}$</td>
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<td>2</td>
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</table>

Examples: $K_4 = \{1, 2^1\}$, $K_{3,3} = \{1, 2^1, 2^2, 3\}$, $Q_3 = \{1, 2^1\}$, $F010A = \{2^1, 3\}$, $F014A = \{1, 4^1\}$, $F016A = \{1, 2^1\}$, $F018A = \{1, 2^1, 2^2, 3\}$, $F020A = \{1, 2^1\}$, $F020B = \{2^1, 2^2, 3\}$. 
The order of automorphisms fixing a vertex in a cubic symmetric graphs

The structure of vertex stabilizers in cubic symmetric graphs implies that only automorphisms of order 1, 2, 3, 4 and 6 can fix a vertex.

<table>
<thead>
<tr>
<th>$s$</th>
<th>$\text{Aut}(X)_v$</th>
<th>$\text{Aut}(X)_e$</th>
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<td>$\mathbb{Z}_2$</td>
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<tr>
<td>2</td>
<td>$S_3$</td>
<td>$\mathbb{Z}_2^2$ or $\mathbb{Z}_4$</td>
</tr>
<tr>
<td>3</td>
<td>$S_3 \times \mathbb{Z}_2$</td>
<td>$D_8$</td>
</tr>
<tr>
<td>4</td>
<td>$S_4$</td>
<td>$D_{16}$ or $QD_{16}$</td>
</tr>
<tr>
<td>5</td>
<td>$S_4 \times \mathbb{Z}_2$</td>
<td>$(D_8 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$</td>
</tr>
</tbody>
</table>
Proposition (KK, DM, 2017)

Let $X$ be a cubic symmetric graph and let $\alpha \in \text{Aut}(X)$ be an automorphism of $X$ fixing a vertex.

(i) If $\alpha$ is of order 3 or 6 then the only $\alpha$-rigid cells are isolated vertices.

(ii) If $\alpha$ is of order 4 then the only possible $\alpha$-rigid cells are $I$-trees.
Rigid cells of involutions

We will say that an involution in the automorphism group of a cubic symmetric graph is

- an $I$-involution if all of its rigid cells are isomorphic to the $I$-tree,
- a $Y$-involution if all of its rigid cells are isomorphic to the $Y$-tree,
- an $H$-involution if all of its rigid cells are isomorphic to the $H$-tree,
- an $A$-involution if all of its rigid cells are isomorphic to the $A$-tree,
- an $M$-involution if it admits non-isomorphic rigid cells (and thus its rigid cells are of \textit{mixed} structure), and
- an $S$-involution if it is semiregular.
Any involution in $X$ is either conjugate to the involution 
$(0)(0')(1)(4)(1' 2)(3 4')(2' 3')$ with one rigid cell isomorphic to the $Y$-tree, or to the involution $(0)(0')(1 4)(1' 3')(1' 4')(2 3)$ with one rigid cell isomorphic to the $I$-tree.
The Petersen graph - an example of type \( \{2^1, 3\} \)

Any involution in \( X \) is either conjugate to the involution 
\((0)(0')(1)(4)(1' 2)(3 4')(2' 3')\) with one rigid cell isomorphic to the 
\( Y \)-tree, or to the involution 
\((0)(0')(1 4)(1' 3')(1' 4')(2 3)\) with one rigid 
cell isomorphic to the \( I \)-tree.

Proposition (KK, DM, 2017)

Every cubic 3-regular graph admitting a 2-regular subgroup has two 
conjugacy classes of non-semiregular involutions, one consisting of 
\( I \)-involutions, and one consisting of \( Y \)-involutions.
There exist two conjugacy classes of involutions in $\text{Aut}(X)$, one consisting of $S$-involutions and one consisting of $H$-involutions. In particular, any non-semiregular involution in $X$ is conjugate to the involution $(0)(1)(2)(3\ 11)(4\ 12)(5\ 7)(6)(8\ 10)(9)(13)$ with one rigid cell which is isomorphic to the $H$-tree.

\begin{center}
\begin{tikzpicture}
\tikzset{VertexStyle/.style = {shape = circle, draw = black, fill = black!80, minimum size = 0.5cm}}
\tikzset{EdgeStyle/.style = {line width = 0.5pt}}
\begin{scope}[xshift = 0cm, yshift = 0cm]
\node[VertexStyle] (1) at (0,0) {1};
\node[VertexStyle] (2) at (1,1.732) {2};
\node[VertexStyle] (3) at (2,0) {3};
\node[VertexStyle] (4) at (1,-1.732) {4};
\node[VertexStyle] (5) at (2,-0.289) {5};
\node[VertexStyle] (6) at (2,-1.732) {6};
\node[VertexStyle] (7) at (1,-3.464) {7};
\node[VertexStyle] (8) at (0,-2.732) {8};
\node[VertexStyle] (9) at (-1,-1.732) {9};
\node[VertexStyle] (10) at (-2,0) {10};
\node[VertexStyle] (11) at (-1,1.732) {11};
\node[VertexStyle] (12) at (-2,-0.289) {12};
\node[VertexStyle] (13) at (-2,-1.732) {13};
\node[VertexStyle] (0) at (0,0) {0};
\draw[EdgeStyle] (1) -- (2);
\draw[EdgeStyle] (2) -- (3);
\draw[EdgeStyle] (3) -- (6);
\draw[EdgeStyle] (6) -- (9);
\draw[EdgeStyle] (9) -- (8);
\draw[EdgeStyle] (8) -- (7);
\draw[EdgeStyle] (7) -- (10);
\draw[EdgeStyle] (10) -- (11);
\draw[EdgeStyle] (11) -- (12);
\draw[EdgeStyle] (12) -- (13);
\draw[EdgeStyle] (13) -- (1);
\end{scope}
\end{tikzpicture}
\end{center}
There exist two conjugacy classes of involutions in $\text{Aut}(X)$, one consisting of $S$-involutions and one consisting of $H$-involutions. In particular, any non-semiregular involution in $X$ is conjugate to the involution $(0)(1)(2)(3\ 11)(4\ 12)(5\ 7)(6)(8\ 10)(9)(13)$ with one rigid cell which is isomorphic to the $H$-tree.

**Proposition (KK, DM, 2017)**

In every cubic 4-regular graph non-semiregular involutions are $H$-involutions.
The dodecahedron - an example of type $\{1, 2^1\}$

There exist three conjugacy classes of involutions in $\text{Aut}(X)$, two consisting of $S$-involutions and one consisting of $I$-involutions. Any non-semiregular involution in $X$ is conjugate to the involution

$$(0)(1\ 9)(2\ 8)(3\ 7)(4\ 6)(5)(0')(1'\ 9')(2'\ 8')(3'\ 7')(4'\ 6')(5')$$

which has two rigid cells, both isomorphic to the $I$-tree.
Tutte’s 8-cage - an example of type $\{4^1, 4^2, 5\}$

There exist three conjugacy classes of involutions in $\text{Aut}(X)$, one consisting of $S$-involutions, one consisting of $H$-involutions, and one consisting of $A$-involutions. Any non-semiregular involution in $X$ is either conjugate to the $A$-involution $(0)(1)(2)(6)(10)(22)(23)(24)(28)(29)(3\ 15)(5\ 7)(9\ 11)(13\ 21)(17\ 25)(19\ 27)(4\ 16)(8\ 12)(14\ 20)(18\ 26)$ with one rigid cell which is isomorphic to the $A$-tree, or to the $H$-involution $(0)(1)(2)(10)(23)(29)(3\ 15)(6\ 28)(9\ 11)(22\ 24)(4\ 14)(5\ 27)(7\ 19)(8\ 18)(12\ 26)(13\ 25)(16\ 20)(17\ 21)$ with one rigid cell which is isomorphic to the $H$-tree.
Results and examples

**Proposition (KK, DM, 2017)**

Every cubic 5-regular graph admitting a 4-regular subgroup has two conjugacy classes of non-semiregular involutions, one consisting of $H$-involutions and one consisting of $A$-involutions.
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Proposition (KK, DM, 2017)

Let $X$ be a cubic 3-regular graph and let $\alpha \in \mathcal{I}(X)$ be a non-semiregular involution of $X$. Then the possible $\alpha$-rigid cells are $I$-trees and $Y$-trees, with both types of cells possibly occurring simultaneously only when the graph $X$ is of type $\{3\}$.

Proposition (KK, DM, 2017)

Let $X$ be a cubic 5-regular graph and let $\alpha \in \mathcal{I}(X)$ be a non-semiregular involution of $X$. Then the only possible $\alpha$-rigid cells are $H$-trees and $A$-trees, with both types of cells possibly occurring simultaneously only when the graph $X$ is of type $\{5\}$.
Theorem (Conder, Hujdurović, KK, DM, 2018)

Let $X$ be a cubic symmetric graph of type \{s\}, where $s \in \{3, 5\}$, and of order $2n$, where $n$ is odd. Then $\mathcal{I}(X)$, the set of all involutions in $\text{Aut}(X)$ which fix some vertex of $X$, is a conjugacy class. In particular, every involution $\alpha \in \mathcal{I}(X)$ is an $M$-involution, and

- if $s = 3$ then $\alpha$ has rigid cells isomorphic to the $I$-tree as well as rigid cells isomorphic to the $Y$-tree, and
- if $s = 5$ then $\alpha$ has rigid cells isomorphic to the $H$-tree as well as rigid cells isomorphic to the $A$-tree.
There exist two conjugacy classes of involutions in $\text{Aut}(X)$, one consisting of $S$-involutions and one consisting of $M$-involutions having rigid cells isomorphic to the $I$-tree as well as rigid cells isomorphic to the $Y$-tree. Every $M$-involution is conjugate to

$$u_0^0, u_0^4, u_1^0, u_1^7, u_2^4, u_3^9, u_3^{10}, u_4^1, u_4^2, u_4^7, u_4^{10}, u_5^1, u_6^7.$$  

with two rigid cells isomorphic to the $Y$-tree and three rigid cells isomorphic to the $I$-tree.
Using Magma one can see that there exists a cubic symmetric graph of type \{3\} and of order 39,916,800 with the automorphism group isomorphic to the symmetric group $S_{12}$. For this graph, there are two conjugacy classes of involutions with fixed points. In one of these two classes each involution fixes 15360 vertices, which partition into 3840 rigid cells isomorphic to the $Y$-tree (and no rigid cell isomorphic to the $I$-tree), while in the other class, each involution fixes 2304 vertices, which partition into 1152 rigid cells isomorphic to the $I$-tree (and no rigid cell isomorphic to the $Y$-tree).
Example of a type \( \{5\} \) graph of order 0 \( \text{mod} \ 4 \)

Using Magma one can see that there exists a cubic symmetric graph of type \( \{5\} \) and of order 50, 685, 458, 503, 680, 000 with the automorphism group isomorphic to the symmetric group \( S_{20} \). For this graph, there are two conjugacy classes of involutions with fixed points, one consisting of involutions with all rigid cells isomorphic to the \( A \)-tree and one consisting of involutions with all rigid cells isomorphic to the \( H \)-tree.
Let $G$ be a finite group with no normal subgroup of index 2 and let $P$ be a Sylow 2-subgroup of $G$. If $R < P$ with $|P : R| = 2$, then every involution of $G$ is conjugate to an involution in $R$. 

Gorenstein, Exercise 7.3(i), 1980
Lemma (Conder, Hujdurović, KK, DM, 2018)
Let $X$ be a bipartite cubic symmetric graph of order $2n$, where $n$ is odd, and either of type $\{3\}$ or of type $\{5\}$, and let $G$ be the index 2 subgroup in $\text{Aut}(X)$ fixing the two bipartite subsets of the vertex set $V(X)$. Then $G$ has no index 2 subgroup.
Let $u, v \in V(X)$ such that $e = uv$ is an edge of $X$. Let the vertices of $X$ in the neighborhood $N(e)$ of $e = uv$ be denoted in such a way that:

\[
\begin{align*}
N(u) &= \{v, 1, 3\}, & N(v) &= \{u, 2, 4\}, \\
N(1) &= \{u, 1', 5'\}, & N(2) &= \{v, 2', 6'\}, \\
N(3) &= \{u, 3', 7'\}, & N(4) &= \{v, 4', 8'\}.
\end{align*}
\]

Djoković and Miller proved that the edge-stabilizer $\text{Aut}(X)_e$ of $e$ is isomorphic to $(D_8 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$. Since $X$ is of order $2 \pmod{4}$ this implies that $\text{Aut}(X)_e$ is a Sylow 2-subgroup of $\text{Aut}(X)$. We distinguish two cases depending on whether $X$ is bipartite or not.
Case 1: $X$ is non-bipartite.

Since $X$ is non-bipartite and of type $\{5\}$ there is no subgroup of index 2 in $\text{Aut}(X)$, and Gorenstain’s exercise can be applied. In particular, we need to prove that there exists an index 2 subgroup in $\text{Aut}(X)_e$ all of whose involutions are of ‘the same structure’.

There exist $\rho, \tau \in \text{Aut}(X)_e$ whose restrictions to $\mathcal{N}(e)$ are the following permutations

$$\bar{\rho} = (u \; v)(1 \; 2 \; 3 \; 4)(1' \; 2' \; 3' \; 4' \; 5' \; 6' \; 7' \; 8')$$

$$\bar{\tau} = (u)(v)(1)(3)(2 \; 4)(1')(5')(2' \; 4')(3' \; 7')(6' \; 8').$$

A short calculation shows that $\bar{\tau}\bar{\rho}\bar{\tau} = \bar{\rho}^3$, and hence there exists $H \leq \text{Aut}(X)_e$ such that its restrictions $\bar{H}$ to $\mathcal{N}(e)$ is equal to

$$\bar{H} = \langle \bar{\rho}, \bar{\tau} \mid \bar{\rho}^8 = \bar{\tau}^2 = 1, \bar{\tau}\bar{\rho}\bar{\tau} = \bar{\rho}^3 \rangle,$$

and hence isomorphic to the quasidihedral group $QD_{16}$ (also called semidihedral group) of order 16.
The proof of the main theorem - for type \{5\} graphs

Any element in $\tilde{H}$ is clearly of the form $\tilde{\tau}^i \tilde{\rho}^j$ where $i \in \mathbb{Z}_2$ and $j \in \mathbb{Z}_8$, and \{\tilde{\tau}, \tilde{\rho}^4, \tilde{\tau} \tilde{\rho}^2, \tilde{\tau} \tilde{\rho}^4, \tilde{\tau} \tilde{\rho}^6\} is the set of involutions in $\tilde{H}$. Further, we have

\[
\begin{align*}
\tilde{\rho}^4 &= (u)(v)(1)(2)(3)(4)(1' 5')(2' 6')(3' 7')(4' 8'), \\
\tilde{\tau} \tilde{\rho}^2 &= (u)(v)(1 3)(2)(4)(1' 7')(2')(3' 5')(4' 8')(6'), \\
\tilde{\tau} \tilde{\rho}^4 &= (u)(v)(1)(2 4)(3)(1' 5')(2' 8')(3')(4' 6')(7'), \\
\tilde{\tau} \tilde{\rho}^6 &= (u)(v)(1 3)(2)(4)(1' 3')(2' 6')(4')(5' 7')(8'),
\end{align*}
\]

implying that the rigid cells of the involutions in $\tilde{H}$ are all isomorphic to the $H$-tree. Since $|\text{Aut}(X)_e| = 32$ and $|\tilde{H}| = 16$ we can conclude that either $H$ is an index 2 subgroup of $\text{Aut}(X)_e$ or $H = \text{Aut}(X)_e$. In the first case Gorenstain’s exercise implies that any involution in $\text{Aut}(X)$ is conjugate to an involution in $H$, and thus since there exists an involution in $\text{Aut}(X)$ with a rigid cell isomorphic to the $A$-tree we can conclude that any involution in $\text{Aut}(X)$ has rigid cells isomorphic to the $H$-tree as well as rigid cells isomorphic to the $A$-tree. In the second case the existence of an involution in $\text{Aut}(X)$ with a rigid cell isomorphic to the $A$-tree gives the same conclusion since stabilizers are conjugate subgroups of $\text{Aut}(X)$.
Case 2: \( X \) is bipartite.

Since \( X \) is bipartite and of type \( \{5\} \) there exists a subgroup \( G \) of index 2 in \( \text{Aut}(X) \) such that the stabilizer of the arc \((u, \nu)\) is contained in \( G \). Clearly the stabilizer of the arc \((u, \nu)\) is a Sylow 2-subgroup \( S \) of \( G \) isomorphic to \( D_8 \times \mathbb{Z}_2 \).

There exist \( \alpha, \beta \in \text{Aut}(X)_e \) whose restrictions to \( N(e) \) are the following permutations

\[
\bar{\alpha} = (u)(\nu)(1)(2\ 4)(3)(1')(2'\ 4')(3'\ 7')(5')(6'\ 8')
\]

\[
\bar{\beta} = (u)(\nu)(1\ 3)(2)(4)(1'\ 3')(2')(4'\ 8')(5'\ 7')(6').
\]

A short calculation shows that \( \bar{\alpha}\bar{\beta} \) is of order 4, and hence there exists \( H \leq \text{Aut}(X)_e \) such that its restrictions \( \bar{H} \) to \( N(e) \) is equal to

\[
\bar{H} = \langle \bar{\alpha}, \bar{\beta} | \bar{\alpha}^2 = \bar{\beta}^2 = 1, (\bar{\alpha}\bar{\beta})^4 = 1 \rangle \cong D_8.
\]
Note that any involution in $\bar{H}$ has rigid cells isomorphic to the $H$-tree. Since $|S| = 16$ and $|\bar{H}| = 8$ we can conclude that either $H$ is an index 2 subgroup of $S$ or $H = S$. In the first case, since, by Lemma, there is no index 2 subgroup of $G$, Gorenstain’s exercise implies that any involution in $G$ is conjugate to an involution in $H$, and thus since there exists an involution in $G$ with a rigid cell isomorphic to the $A$-tree we can conclude that any involution in $\text{Aut}(X)$ has rigid cells isomorphic to the $H$-tree as well as rigid cells isomorphic to the $A$-tree. In the second case the existence of an involution in $G$ with a rigid cell isomorphic to the $A$-tree gives the same conclusion since stabilizers are conjugate subgroups of $\text{Aut}(X)$. □
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Portorož, Slovenia

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Colva Roney-Dougal, University of St Andrews, UK

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The conference will be held at UP FAMNIT in Koper, Slovenia.
Dates: 28 May - June 1, 2018
THANK YOU!

HVALA!