The directed genus of some digraphs

Rong-Xia Hao

Department of Mathematics, Beijing Jiaotong University

Jan.28.2018-Feb.2.2018
1. Terms and Background
2. Joint Tree Method
3. The Genus of Some Digraphs
A digraph $D$ is called **Eulerian** if it is connected and the indegree equals to the outdegree at each vertex.

A surface is a compact 2-manifold without boundary. The orientable surface $S_h$ of genus $h$—the sphere with $h$ handles added.

A graph is said to be **embedded in a surface** $S$ if it is drawn in $S$ so that edges intersect only at their common end vertices and each region is homeomorphic to an open disk.
A digraph $D$ is called **Eulerian** if it is connected and the indegree equals to the outdegree at each vertex.

A **surface** is a compact 2-manifold without boundary. The **orientable surface** $S_h$ of genus $h$—the sphere with $h$ handles added. A graph is said to be **embedded in a surface $S$** if it is drawn in $S$ so that edges intersect only at their common end vertices and each region is homeomorphic to an open disk.
A Steiner triple system of order \( n \), STS\((n)\), is a pair \((V, B)\), where \( V \) is a set of \( n \) points and \( B \) is a collection of triples, also called blocks, taken from \( V \) and such that every pair of distinct vertices from \( V \) appears in precisely one block.

An embedding of a Steiner triple system \( B \) in the orientable surface is an embedding of the complete graph \( K_n \) on the vertex set \( V \) such that each block of \( B \) appearing on the surface as a triangle bounds a region of the embedding.
A Steiner triple system of order $n$, STS($n$), is a pair $(V, B)$, where $V$ is a set of $n$ points and $B$ is a collection of triples, also called blocks, taken from $V$ and such that every pair of distinct vertices from $V$ appears in precisely one block.

An embedding of a Steiner triple system $B$ in the orientable surface is an embedding of the complete graph $K_n$ on the vertex set $V$ such that each block of $B$ appearing on the surface as a triangle bounds a region of the embedding.
A **2-colorable embedding** of a graph $G$ is an embedding which admits a 2-coloring of regions such that no two regions of the same color share an edge. Two colors always mean black and white.

For a given 2-colorable triangular embedding of $K_n$ in the orientable surface, the faces in each color class form a steiner triple system of order $n$, one is given by the white faces which is isomorphic to $(V, B)$, while the other generated by the black faces is isomorphic to $(V', B')$ as a STS. Then we say these two STSs are **bi-embedded** and the pair $B$ and $B'$ is **bi-embeddable** in this orientable surface.
An (directed) embedding of a digraph $D$ in the orientable surface $S$ means that it is the embedding of the underlying graph and the directions of arcs are consistent in each region boundary.

An oriented embedding of $D$ on plane

(1) $D$

(2) An oriented embedding of $D$ on plane
A rotation at a vertex $v$ of an Eulerian digraph $D$—a ordering cycle of all the arcs incident with $v$ such that the in-arcs and out-arcs at $v$ alternate. A rotation system (A rotation scheme) of an Eulerian digraph $D$—A list of rotations, one for each vertex. Two directed embedding of an Eulerian digraph $D$ are equivalent (or, informally, the same) if they have the same rotation system.

$$\rho(v_1) = (b, a, e, d), \rho(v_2) = (d, c), \rho(v_3) = (c, e, f, b), \rho(v_4) = (f, a).$$
A rotation at a vertex $v$ of an Eulerian digraph $D$—a ordering cycle of all the arcs incident with $v$ such that the in-arcs and out-arcs at $v$ alternate.

A rotation system (A rotation scheme) of an Eulerian digraph $D$—A list of rotations, one for each vertex. Two directed embedding of an Eulerian digraph $D$ are equivalent (or, informally, the same) if they have the same rotation system.

$$
\rho(v_1) = (b, a, e, d), \rho(v_2) = (d, c), \rho(v_3) = (c, e, f, b), \rho(v_4) = (f, a).
$$
There exists a one to one correspondence between the set of all rotation systems and the set of (directed) embeddings of an Eulerian digraph $D$. 
The (directed) genus of a digraph $D$, denoted $\gamma(D)$ and the maximum (directed) genus of $D$ denoted $\gamma_M(D)$, are the minimum value and maximum value of $p$, respectively, for which the digraph $D$ has a directed embedding into a surface of genus $p$.

$g_i(D)$—the number of different embeddings of $D$ in the surface with (directed) genus $i$, the sequence $g_0(D), g_1(D), g_2(D), \cdots$ is called the genus (embedding) distribution of $D$.

The genus polynomial of a digraph $D$— $f_D(x) = \sum_{i=0}^{\infty} g_i(D)x^i$
The (directed) genus of a digraph $D$, denoted $\gamma(D)$ and the maximum (directed) genus of $D$ denoted $\gamma_M(D)$, are the minimum value and maximum value of $p$, respectively, for which the digraph $D$ has a directed embedding into a surface of genus $p$.

$g_i(D)$—the number of different embeddings of $D$ in the surface with (directed) genus $i$, the sequence $g_0(D), g_1(D), g_2(D), \cdots$ is called the genus (embedding) distribution of $D$.

The genus polynomial of a digraph $D$—

$$f_D(x) = \sum_{i=0}^{\infty} g_i(D)x^i$$
There are rich results about genus distributions of graphs, but little is known about the genus of a digraph.

<table>
<thead>
<tr>
<th>For examples:</th>
</tr>
</thead>
</table>
Embedding of eulerian digraphs are studied systematically beginning from Bonnington, Conder, Morton and McKenna. Analogous to the characterizations of graphs in terms of graph minors, Bonnington, Hartsfield, Širáň obtained the obstructions for directed embedding of digraphs and proved Kuratowski-type theorems for directed embeddings in the plane.

Theorem [Bonnington et al. 2002]
The spoke digraph $SD_{2k+1}$ on $n = 2k + 1$ vertices has maximum genus $k$ and minimum genus $k - 1$. 
Joint tree method for computing the genus polynomial

G. Ringel

An orientable surface can be written as a cyclic sequence of letters (i.e., a string of letters) such that each letter appears exactly twice and the two occurrences of each letter with distinct powers "+" (which is always omitted) and "−".

The algebraic representation of each orientable surface is equivalent to only one of the following canonical forms:

\[ O_i = \begin{cases} 
  (a_0a_0^-), & \text{if the surface is sphere} \\
  \left( \prod_{k=1}^{i} a_k b_k a_k^- b_k^- \right), & \text{if the genus of a surface is } i 
\end{cases} \]

Let $D$ be a digraph and $T$ be a spanning tree of $D$. For each non-tree arc $e$, $e$ is split into two semi-arcs $e^+$ and $e^-$. It is obvious that the digraph obtained by splitting all non-tree arcs is a tree, which is called a joint tree $\tilde{T}$. (We shall use $e$ for $e^+$ for the sake of convenience without confusion.

For a joint tree $\tilde{T}$ of $D$, a cyclic sequence of all letters of semi-arcs along the clockwise (or anticlockwise) rotation is an algebraic representation of the embedded surface of $D$. It is known that the genus distribution is independent of the choice of a tree. Let $S$ be the set of all surfaces, an equivalence, denoted by $\sim$, between two surfaces is introduced by the following three elementary transformation (or operations):

OP1 $$(A) \sim (Aee^-), \text{ where } A \in S, A \neq \emptyset \text{ and } e \notin A;$$
OP2 $$(Ae_1e_2Be^-_2e^-_1) \sim (AeBe^-);$$
OP3 $$(AB) \sim (Aee^-B), \text{ where } AB \in S, AB \neq \emptyset \text{ and } e \notin AB.$$

Where, $A$ and $B$ are sections of successive letters in cyclic orders. $e, e_1$ and $e_2$ are distinct letters.
Fig. 3 shows the digraph $D$. The di-path $v_4v_3v_1v_2$ is chosen as a spanning tree $T$ with $a$, $b$ and $c$ non-tree arcs. Then the corresponding embedding surfaces are

(1) $(abcc^-b^-a^-)$, (2) $(a^-cbac^-b^-)$, (3) $(a^-b^-c^-abc)$, (4) $(a^-b^-c^-cba)$. 
Fig. 3

(3) \( (a^*b^*c^*a^*) \sim (a^*a^*), \ g = 0; \)
(2) \( (a^*-cbaca^*-b^*) \sim (a^*-bab^*) \sim (a^*-b^*-ab), \ g = 1; \)
(3) \( (a^*-b^*-c^*-abc) \sim (a^*-b^*-ab), \ g = 1; \)
(4) \( (a^*-b^*-c^*-cba) \sim (a^*a^*), \ g = 0. \)

As a consequence, the genus polynomial of this digraph \( D \) on orientable surfaces is \( f_D(x) = 2 + 2x \).
The genus polynomial of the digraph $D$ on orientable surfaces is $f_D(x) = 2 + 2x$. 
By using joint tree method, we got genus distributions of some digraphs. For example:
Antiladder $DL_n$

Let $C$ be the di-circuit with $4n + 4$ vertices, say $u, u_1, u_2, \ldots, u_{2n-1}, u_{2n}, u', v, v_1, v_2, \ldots, v_{2n-1}, v_{2n}, v'$ along the orientation of $C$. An antiladder denoted by $DL_n$ is a digraph which is obtained from $C$ by adding $2n + 1$ pairs of diagonal arcs $a_i^1 = \langle u_i, v_{i+1} \rangle$, $a_i^2 = \langle v_{i+1}, u_i \rangle$, $b_i^1 = \langle u_{i+1}, v_i \rangle$, $b_i^2 = \langle v_i, u_{i+1} \rangle$ for each odd $i$ and arcs $b = \langle u, v \rangle$, $c = \langle v, u \rangle$. $DL_n$ is a non-planar digraph for $n \geq 2$.

$DL_n$ is shown in Figure

![Figure: DL_n]
**Theorem [Hao and Liu 2008]**

Let $g_i(DL_n)$ be the number of different embeddings of directed antiladder $DL_n$ into orientable surfaces of genus $i$, then,

$$g_i(DL_n) = 2g_{(i-1)1}(n) + 2g_{i2}(n).$$

<table>
<thead>
<tr>
<th>$g_{ij}(n)$</th>
<th>Case</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>$4r_{(i-2)1} + 8r_{(i-1)2} + 2r_{i2} + 2r_{i3}$</td>
<td>If $j = 1$ and $0 \leq i \leq 2n$;</td>
<td></td>
</tr>
<tr>
<td>$4r_{(i-2)2} + r_{(i-1)1} + 4r_{(i-2)1} + r_{i3} + 4r_{(i-1)3} + 2r_{(i-1)2}$</td>
<td>If $j = 2$ and $0 \leq i \leq 2n+1$;</td>
<td></td>
</tr>
<tr>
<td>$4r_{(i-2)3} + 2r_{(i-1)2} + 8r_{(i-2)2} + 2r_{(i-2)1}$</td>
<td>If $j = 3$ and $0 \leq i \leq 2n+1$.</td>
<td></td>
</tr>
</tbody>
</table>

Where $g_{ij}(n-1) = r_{ij}$.

By applying this theorem, the genus polynomials \( f_D(x) = \sum_{i=0}^{\infty} g_i(D)x^i \) of \( D = DL_n \) for a given \( n \) can be obtained. For examples \( f_D(x) \) for \( n = 0, 1, 2, 3, 4 \) are calculated as follows. (Let \( g_{0j}(0) = 1, g_{ij}(0) = 0 \) for all \( i \neq 0 \) and \( g_{ij}(n) = 0 \) for all \( i < 0 \).)

\[
\begin{align*}
  f_{DL_0}(x) &= 2 + 2x; \\
  f_{DL_1}(x) &= 2 + 22x + 32x^2 + 8x^3; \\
  f_{DL_2}(x) &= 20x + 180x^2 + 504x^3 + 288x^4 + 32x^5; \\
  f_{DL_3}(x) &= 8x + 216x^2 + 1712x^3 + 5360x^4 + 6912x^5 + 2048x^6 + 128x^7; \\
  f_{DL_4}(x) &= 112x^2 + 2224x^3 + 16576x^4 + 58784x^5 + 99200x^6 + 71936x^7 + 12800x^8 + 512x^9; \\
\end{align*}
\]
Some problems are given by Bonnington et al.

Problem 1—Is the genus distribution of an embeddable digraph always unimodal, as is conjectured to be the case in the study of undirected graphs by Gross et al.?

A real sequence \( a_0, a_1, \ldots, a_n \) is called unimodal if for some number \( m \) such that \( 0 \leq m \leq n \), we have \( a_0 \leq a_1 \leq \ldots \leq a_m \geq a_{m+1} \geq \ldots \geq a_n \). Moreover, if for every \( j \) such that \( 1 \leq j \leq n - 1 \), we have \( a_j^2 \geq a_{j-1}a_{j+1} \), then the sequence is called log-concave or strongly unimodal. A strongly unimodal sequence is unimodal.

Problem 2—Which tournaments on \( n \) vertices have directed genus \( \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil \), the genus of \( K_n \)?

Chen, Gross and Hu gave a splitting theorem for digraph embedding distributions that is analogous to the splitting theorem for (undirected) graph genus distributions.

**Theorem [Chen Gross and Hu 2014]**

The genus distribution of a 4-regular outer planar digraph is strongly unimodal.

Problem 2—Which tournaments on $n$ vertices have directed genus $\left\lceil \frac{(n-3)(n-4)}{12} \right\rceil$, the genus of $K_n$.


Theorem (Hao 2018)

There exists a tournament on $n$ vertices having directed genus $\left\lceil \frac{(n-3)(n-4)}{12} \right\rceil$ if and only if $n \equiv 3$ or $7 \pmod{12}$.
Problem 2—Which tournaments on $n$ vertices have directed genus $\left\lceil \frac{(n-3)(n-4)}{12} \right\rceil$, the genus of $K_n$.


**Theorem (Hao 2018)**

There exists a tournament on $n$ vertices having directed genus $\left\lceil \frac{(n-3)(n-4)}{12} \right\rceil$ if and only if $n \equiv 3$ or 7 (mod 12).
The genus of the tournament

Theorem (Hao 2018)
The following two conditions are equivalent.

1. There exists a pair of bi-embedded Steiner triple systems (STSs) $(V, B)$ and $(V', B')$ with $|V| = |V'| = n$ in an orientable surface.

2. There is a tournament on $n$ vertices which has a directed embedding on the orientable surface of genus $\lceil \frac{(n-3)(n-4)}{12} \rceil$, and the set of faces is $B$, the set of antifaces is $B'$. 

Rong-Xia Hao ()
The directed genus of some digraphs
25 / 26
Thank You!