Diameter Bounds on Geometric Distance-regular Graphs

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Outline

1 Geometric distance-regular graphs

2 Motivation

3 Diameter bound
Let $\Gamma = (V(\Gamma), E(\Gamma))$ be a finite connected undirected graph.

- **Distance** $d(x, y)$
- **Diameter** $D = D(\Gamma) := \max\{d(x, y) \mid x, y \in V(\Gamma)\}$.
- For each $0 \leq i \leq D(\Gamma)$, define the $i$-th distance matrix $A_i$ by
  
  $$(A_i)_{(x,y)} := \begin{cases} 
  1 & \text{if } d(x, y) = i \\
  0 & \text{otherwise}
  \end{cases}.$$

- The eigenvalues of $\Gamma$ are the eigenvalues of $A_1$. 
Distance-Regular Graph

A connected graph $\Gamma$ is called a distance-regular graph (DRG) if for each $1 \leq i \leq D(\Gamma)$, there exist constants $a_i, b_i, c_i$ such that

$$A_1 A_i = b_{i-1} A_{i-1} + a_i A_i + c_{i+1} A_{i+1}.$$

$A_i$ : the $i$-th distance matrix where $(A_i)_{(x,y)} := \begin{cases} 1 & \text{if } d(x, y) = i \\ 0 & \text{otherwise} \end{cases}$. 

\[(c_1, a_1, b_1) = (1, 0, 2) \quad (c_2, a_2, b_2) = (2, 0, 1) \quad (c_3, a_3, b_3) = (3, 0, 0)\]
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$$A_1^2 = 3I + 2A_2 \Rightarrow A_2 = \frac{1}{2} (A_1^2 - 3I)$$

$$A_1 A_2 = 2A_1 + 3A_3 \Rightarrow A_3 = \frac{1}{6} (A_1^3 - 7A_1)$$
Let $\Gamma$ be a distance-regular graph.

1. $\Gamma$ is a regular graph with valency $k := b_0$. ($k = c_i + a_i + b_i$)

2. The array $\iota(\Gamma) = \{b_0, b_1, \ldots, b_{D-1}; c_1, c_2, \ldots, c_D\}$ is called the \textbf{intersection array} of $\Gamma$.

3. $\iota(\Gamma) = \{3, 2, 1; 1, 2, 3\}$
Let $\Gamma$ be a DRG with valency $k$ and smallest eigenvalue $\theta_{\text{min}}$.

1. $C$ : clique

2. $|C| \leq 1 - \frac{k}{\theta_{\text{min}}}$

3. $C$ : Delsarte clique if $|C| = 1 - \frac{k}{\theta_{\text{min}}}$
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4. If $\Gamma$ is bipartite then each clique $C$ in $\Gamma$ satisfies

$$|C| = 1 - \frac{k}{\theta_{\text{min}}} = 1 - \frac{k}{-k} = 2.$$
A DRG $\Gamma$ is called \textit{geometric w.r.t. $C$} if $\exists$ a set of Delsarte cliques $C$ such that each edge lies in exactly one clique in $C$ (Delsarte 1973, Godsil 1993).

- Examples of geometric DRG include bipartite graphs, Johnson graph, Grassmann graph, Hamming graphs, Dual polar graphs, bilinear forms graphs ... 

- Hamming Graph $H(D, q)$

$$k = D(q - 1), \quad \theta_{\min} = -D, \quad |C| = 1 - \frac{k}{\theta_{\min}} = q$$
Geometric distance-regular graphs

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- A partial linear space of order $(s, t)$ is a point-line incidence structure s.t. every line has exactly $s + 1$ pts, every point lies in exactly $t + 1$ lines and any two distinct lines meet at at most one pt.

- Geometric DRG is the point graph of a partial linear space of order $\left( \frac{k}{|\theta_D|}, |\theta_D| - 1 \right)$. 

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$$(C_5 : 2, \frac{-1 \pm \sqrt{5}}{2}) \text{ and } |C| = 1 + \frac{k}{|\theta_D|}$$
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$K_{2,2,2,2}$

- eigenvalues $\{6^1, 0^4, (-2)^3\}$
- $|C| = 1 + 6/| - 2| = 4$
- each edge lies in exactly $n = 4$ Delsarte cliques

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(3) Set $C$ for geometric DRG does not need to be unique.
Johnson Graph $J(2s, s)$ ($s \geq 2$)

- **Vertex set:** \( \left[ \begin{array}{c} X \\ s \end{array} \right] \) where \(|X| = 2s\)
- **Distance:** \(d(\alpha, \beta) = 1\) iff \(|\alpha \cap \beta| = s - 1\)
Johnson Graph $J(2s, s) \ (s \geq 2)$

- vertex set: $\binom{X}{s}$ where $|X| = 2s$
- $d(\alpha, \beta) = 1$ iff $|\alpha \cap \beta| = s - 1$
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For any $\gamma \in \binom{X}{s-1}$ and for any $\delta \in \binom{X}{s+1}$, we define

$$\gamma^+ := \left\{ \alpha \in \binom{X}{s} \mid \gamma \subseteq \alpha \right\} \quad \text{and} \quad \delta^- := \left\{ \alpha \in \binom{X}{s} \mid \alpha \subseteq \delta \right\}.$$

Then both $\gamma^+$ and $\delta^-$ induce cliques of size $s + 1$, and thus these are Delsarte cliques in $J(2s, s)$. Let

$$C^+ := \left\{ \gamma^+ \mid \gamma \in \binom{X}{s-1} \right\} \quad \text{and} \quad C^- := \left\{ \delta^- \mid \delta \in \binom{X}{s+1} \right\}.$$
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Geometric DRG with a fixed smallest eigenvalue

- (Neumaier 1979) Except for a finite number of graphs, all geometric SRGs with a given smallest eigenvalue \(-m\) are either Latin square graphs or Steiner graphs.
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- Koolen and Bang (2010): Fix an integer $m \geq 2$. Then there are only finitely many coconnected non-geometric DRG with $\theta_{\text{min}} \geq -m$ and $c_2 \geq 2$. 

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There are no geometric DRGs with $\theta_{\text{min}} = -2$, $D \geq 3$ and $c_2 \geq 2$ (Blokhuis and Brouwer 1997).
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- There are no geometric DRGs with $\theta_{\text{min}} = -2$, $D \geq 3$ and $c_2 \geq 2$ (Blokhuis and Brouwer 1997).

- (Bang 2013) A geometric DRG with $\theta_{\text{min}} = -3$, $D \geq 3$ and $c_2 \geq 2$ satisfies one of the following.
  (a) The Hamming graph $H(3, \alpha + 2)$, where $\alpha \geq 1$.
  (b) The Johnson graph $J(\alpha, 3)$, where $\alpha \geq 6$.
  (c) The collinearity graph of the generalized quadrangle of order $(\alpha + 1, 3)$ deleting the edges in a spread, where $\alpha \in \{2, 4\}$.
  (d) $\nu(\Gamma) = \{3(\alpha + 1), 2(\alpha + 1), \alpha + 2 - \beta; 1, 2, 3\beta\}$, where $\alpha \geq \beta > 1$. 
(i) $k = 3$ and $\Gamma \in \{ \text{the Heawood graph, the Pappus graph, Tutte's 8-cage, the Desargues graph, Tutte's 12-cage, the Foster graph, } K_{3,3}, H(3, 2) \}$

(ii) A SRG with $\left( \frac{(2\alpha-3)(\alpha-2)}{3}, 3\alpha - 9, \alpha, 9 \right)$, where $\alpha \geq 6$ satisfying $\alpha \equiv 0, 2 \pmod{3}$.

(iii) A SRG with $(\alpha^2, 3(\alpha - 1), \alpha, 6)$, where $\alpha \geq 3$.

(iv) The generalized $2D$-gon of order $(s, 2)$, where $(D, s) = (2, 2), (2, 4), (3, 8)$.

(v) One of the two generalized hexagons of order $(2, 2)$ with $\iota(\Gamma) = \{6, 4, 4; 1, 1, 3\}$.

(vi) A generalized octagon of order $(4, 2)$ with $\iota(\Gamma) = \{12, 8, 8, 8; 1, 1, 1, 3\}$.

(vii) The Johnson graph $J(\alpha, 3)$, where $\alpha \geq 6$.

(viii) $D = 3$ and $\iota(\Gamma) = \{3\alpha + 3, 2\alpha + 2, \alpha + 2 - \beta; 1, 2, 3\beta\}$, where $\alpha, \beta \geq 1$.

(ix) The halved Foster graph with $\iota(\Gamma) = \{6, 4, 2, 1; 1, 1, 4, 6\}$. 
Classification of Geometric DRGs with $\theta_{\text{min}} = -3$ - continued

(x) $D = h + 2 \geq 4$ and

$$(c_i, a_i, b_i) = \begin{cases} 
(1, \alpha, 2\alpha + 2) & \text{for } 1 \leq i \leq h \\
(2, 2\alpha + \beta - 1, \alpha - \beta + 2) & \text{for } i = h + 1 \\
(3\beta, 3\alpha - 3\beta + 3, 0) & \text{for } i = h + 2
\end{cases},$$

where $\alpha, \beta \geq 2$.

(xi) $D = h + 2 \geq 3$ and

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where $\alpha, \beta \geq 2$.

(xii) A distance-2 graph of a distance-biregular graph with vertices of valency 3 and

$$(c_i, a_i, b_i) = \begin{cases} 
(1, \alpha, 2\alpha + 2) & \text{for } 1 \leq i \leq h \\
(1, \alpha + 2, 2\alpha) & \text{for } i = h + 1 \\
(4, 2\alpha - 1, \alpha) & \text{for } h + 2 \leq i \leq D - 2 \\
(4, 2\alpha + \beta - 3, \alpha - \beta + 2) & \text{for } i = D - 1 \\
(3\beta, 3\alpha - 3\beta + 3, 0) & \text{for } i = D
\end{cases},$$

where $\alpha \geq 2$ and $\beta \in \{2, 3\}$. 
These satisfy $\overline{A}K_{2,1,1}$.

Note that a distance-regular graph has no induced subgraph $K_{2,1,1}$ iff it is of order $(s,t)$ with $s = a_1 + 1$ and $t = \frac{b_1}{a_1+1}$ (i.e., locally the disjoint union of $t + 1$ cliques of size $s$).

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\end{cases}$$
Now, we consider the case which contains an induced subgraph $K_{2,1,1}$. 
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(Bang 2013): The Johnson graph $J(n,3)$, where $n \geq 6$ is the only geometric DRG satisfying $\theta_{\text{min}} = -3$, $D \geq 3$ and $\exists K_{2,1,1}$. 
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Question

What are the geometric DRGs satisfying $\theta_{\text{min}} = -4$, $D \geq 3$ and $\exists K_{2,1,1}$?
Geometric DRG containing an induced subgraph $K_{2,1,1}$

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**Question**

What are the geometric DRGs satisfying $\theta_{\text{min}} = -4$, $D \geq 3$ and $\exists K_{2,1,1}$?

**Corollary (Bang 2018)**

Suppose that $\Gamma$ is a geometric DRG with $\theta_{\text{min}} \geq -4$ and $D \geq 3$. If $\exists K_{2,1,1}$ then $\Gamma = J(n, D)$ ($n \geq 2D$), where $D \in \{3, 4\}$.
Geometric distance-regular graphs

Motivation

Diameter bound
Let $\Gamma$ be a geometric DRG, and let $C$ be a Delsarte clique.

- $\psi_i = \psi_i(z, C) := |\{ x \in C \mid d(x, z) = i \}|$

- $\tau_i = \tau_i(x, y; C) := |\{ C \in C \mid x \in C \text{ and } d(y, C) = i - 1 \}|$

- $b_i = - (\theta_{\min} + \tau_i) \left(1 - \frac{k}{\theta_{\min}} - \psi_i\right) \quad (1 \leq i \leq D - 1)$

- $c_i = \tau_i \psi_{i-1} \quad (1 \leq i \leq D)$
Let $\Gamma$ be a geometric DRG, and let $C$ be a Delsarte clique, $|C| := s + 1$.

- $\psi_i(z, C) := |\{x \in C \mid d(x, z) = i\}|$
- $\psi_i = \psi_i(z, C) \ \forall i$
- $1 \leq \psi_1 \leq \psi_2 \leq \cdots \leq \psi_{D-1} \leq s$
\[ \psi_i : 1 \leq \psi_1 \leq \psi_2 \leq \ldots \leq \psi_{D-1} \leq s := |C| - 1 \]

\[ \psi_i(z, C) := |\{x \in C \mid d(x, z) = i\}| \]

- \[ 1 = \psi_1 = \psi_2 = s \text{ and } \not\exists K_{2,1,1} \]
- If \( K_{2,1,1} \) exits then \( 2 \leq \psi_1 < \psi_2 < \ldots < \psi_{D-1} \leq s \).

\[ \Rightarrow D \leq s \text{ (Bang-Hiraki-Koolen 2007)} \]
Let $\Gamma$ be a geometric DRG, and let $C$ be a Delsarte clique $|C| := s + 1$.

- $\tau_i(x, y; C) := \left| \{ C \in C \mid x \in C \text{ and } d(y, C) = i - 1 \} \right|

- $\tau_i = \tau_i(x, y; C)$ \quad \forall i$

- $1 = \tau_1 \leq \tau_2 \leq \cdots \leq \tau_D = -\theta_{\min}$
\( \tau_i : 1 = \tau_1 \leq \tau_2 \leq \cdots \leq \tau_D = -\theta_{\min} \)

\[
\tau_i := |\{C \in \mathcal{C} \mid x \in C \text{ and } d(y, C) = i - 1\}| 
\]

\[
1 = \tau_1 < \tau_2 = 2 < \tau_3 = 3 = |\theta_{\min}| 
\]

If \( K_{2,1,1} \) exits then \( 2 \leq \psi_1 \leq \tau_2 < \cdots < \tau_D = m. \)

\[\Rightarrow \quad D \leq m \text{ (Bang 2018)}\]

It is known that if \( c_2 \geq 2 \) then \( D < m^2 \) (Koolen-Bang 2010).
Theorem (Bang 2018)

Fix an integer \( m \geq 2 \). Let \( \Gamma \) be a geometric DRG with \( D \geq 2 \) and \( \theta_{\text{min}} = -m \). If \( K_{2,1,1} \) exists then

\[ D \leq m. \]

In particular, if \( \max\{3, m - 1\} \leq D \leq m \) then \( \Gamma \) is a Johnson graph with \( D = m \).
Theorem (Bang 2018)

Fix an integer $m \geq 2$. Let $\Gamma$ be a geometric DRG with $D \geq 2$ and $\theta_{\min} = -m$. If $K_{2,1,1}$ exits then

$$D \leq m.$$ 

In particular, if $\max\{3, m-1\} \leq D \leq m$ then $\Gamma$ is a Johnson graph with $D = m$.

Corollary (Bang 2018)

Suppose that $\Gamma$ is a geometric DRG with $D \geq 3$ and $\theta_{\min} \geq -4$. If $K_{2,1,1}$ exists, then $\Gamma$ is the Johnson graph $J(n, D)$ ($n \geq 2D$), where $D \in \{3, 4\}$. 
Characterization of Johnson graphs $J(n, D)$ ($n \geq 2D$)

**Theorem (Bang 2018)**

Fix an integer $m \geq 2$. Let $\Gamma$ be a geometric DRG with $D \geq 2$ and $\theta_{\text{min}} = -m$. If $K_{2,1,1}$ exits then

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**Corollary (Bang 2018)**

Suppose that $\Gamma$ is a geometric DRG with $D \geq 3$ and $\theta_{\text{min}} \geq -4$. If $K_{2,1,1}$ exists, then $\Gamma$ is the Johnson graph $J(n, D)$ ($n \geq 2D$), where $D \in \{3, 4\}$.

- **Characterization by parameters**
  - $D = 2$: Connor ($n \geq 9$), Shrikhande ($n \leq 6$), Hoffman, Chang ($n = 7$), three Chang’s graphs ($n = 8$)
    (Many special cases had been settled earlier than Neumaier and Terwilliger)
  - $D = 3$: Bose-Laskar ($n > 16$), Aigner ($n \leq 8$), Rolland ($n \geq 9$)
  - Moon ($n = 9, 10$), Liebler ($11 \leq n \leq 16$)
  - $D > 3$ Moon ($n \in \{2D + 1, 2D + 2, 3D, 3D + 1\}$, $n > 4D$ with $n \geq 20$)
Thank you for your attention!