Recent developments on edge-transitive graphs and maps

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**Edge-transitive graphs**

A graph is edge-transitive (ET) if its automorphism group has a single orbit on edges. Such a graph might or might not be also vertex-transitive (VT), or arc-transitive (AT).

**Examples**

- Cycle graphs $C_n$ are ET, VT and AT
- Complete graphs $K_n$ are ET, VT and AT
- Complete bipartite graphs $K_{m,n}$ are all ET, but are VT (and AT) only when $m = n$
- Semi-symmetric graphs are regular and ET but not VT (e.g. the Folkman and Gray graphs).
- Half-arc-transitive graphs are ET and VT but are not AT (e.g. the Doyle-Holt graph of order 27).
Worthy and unworthy graphs

A graph is called unworthy [thanks to Steve Wilson] if two of its vertices \( v \) and \( w \) have exactly the same neighbours:

In this case, there exists an automorphism of the graph that swaps \( v \) and \( w \) but fixes all others. This can make the automorphism group quite large (but imprimitive).

Examples include the complete bipartite graphs \( K_{m,n} \) for \( \max(m,n) \geq 2 \), with automorphism group \( S_m \times S_n \) or \( S_n \rtimes C_2 \).

If there is no such pair \( \{v, w\} \) then the graph is worthy.
Finding vertex-transitive graphs

It is fairly straightforward to find all small vertex-transitive graphs using the list of all transitive groups of small degree. These groups and graphs are known for degree $n$ up to 47. (For a long time, those of degree 32 were a road-block.)

For a vertex-transitive graph $X$ of order $n$, let $A = \text{Aut}(X)$. Then $A$ acts transitively and faithfully on $V(X)$, and for any vertex $v$, the neighbourhood $X(v)$ is a union of orbits of the vertex-stabiliser $A_v$ on $V(X) \setminus \{v\}$.

Conversely, if $P$ is any transitive group of degree $n$, and $\Delta$ is a union of orbits of the point-stabiliser $P_1$, then the union of the $P$-orbits of the pairs $\{1, \delta\}$ with $\delta \in \Delta \setminus \{1\}$ gives the edge-set of a graph $X$ of order $n$ on which $P$ acts vertex-transitively. (Note: $P$ need not be all of Aut$(X)$.)
What about edge-transitive graphs?

To find all small edge-transitive (ET) graphs, it’s helpful to consider separately the two cases according to whether or not the graph is also vertex-transitive (VT).

It turns out that the latter case helps with the former case when the graph is bipartite.
Case (a): **Edge- but not vertex-transitive graphs**

In this case \( \text{Aut}(X) \) has two orbits on \( V(X) \), say \( A \) and \( B \), with every edge of \( X \) joining a vertex of \( A \) to a vertex of \( B \). Hence in particular, \( X \) is bipartite.

Also \( X \) is locally arc-transitive (i.e. the stabiliser in \( \text{Aut}(X) \) of any vertex \( v \) is transitive on the arcs of the form \((v, w)\)).

Next, we may **suppose \( X \) is worthy** (with no two vertices having the same neighbours), since every unworthy edge-transitive unworthy graph can be constructed as a ‘blow-up’ of a worthy example. [Why/how? See next slide]
‘Blow-ups’ of ET graphs

Let $X$ be a bipartite graph, with parts $A$ and $B$ (say).

Then for any pair $(m,n)$ of positive integers, replace every vertex $a \in A$ by $m$ new vertices $a_1, a_2, \ldots, a_m$ and every vertex $b \in B$ by $n$ new vertices $b_1, b_2, \ldots, b_n$, and replace every edge $\{a,b\}$ by $mn$ edges $\{a_i, b_j\}$ for $1 \leq i \leq m$ and $1 \leq j \leq n$.

This gives the $(m,n)$ blow-up of $X$.

For example, the $(m,n)$ blow-up of $K_2$ is the graph $K_{m,n}$.

If $X$ is edge-transitive, then so are all of its blow-ups. And conversely, if $Y$ is an unworthy ET bipartite graph, then $Y$ is a (unique) blow-up of a worthy ET bipartite graph $X$, each of whose vertices corresponds to a maximal set of vertices of $Y$ having the same neighbourhood (in $Y$).
Moreover, when $X$ is a worthy ET bipartite graph (with parts $A$ and $B$), then $G = \text{Aut}(X)$ acts faithfully and transitively on each of the parts $A$ and $B$ of $X$, and hence we can think of $G$ as a transitive permutation group on $A$ with an auxiliary faithful and transitive action on $B$.

In particular, the neighbours in $B$ of a vertex $a \in A$ form an orbit of $G_a$ in its action on $B$.

This gives a way to find all such graphs: For every transitive group $G$ on a set $A$ of given degree $|A|$, find all faithful transitive actions of $G$ on a set $B$ of given degree $|B|$, and for each one, construct the bipartite graph whose edges are the pairs in an orbit of $G$ on $A \times B$. 
Algorithm

To find all worthy edge-transitive bipartite graphs with parts of sizes $m$ and $n$:

(1) Find all transitive permutation groups of degree $m$
(2) For each such group $P$, find all faithful transitive permutation representations of $P$ of degree $n$
(3) For each such representation, and each orbit of the point-stabiliser $P_1$, check the resulting bipartite graph for connectedness, worthiness, $A$- and $B$-valencies, isomorphism with earlier graphs found, vertex-transitivity, and so on.

Note: Step (2) can be taken by finding all conjugacy classes of core-free subgroups of index $n$ in $P$, and simplified by eliminating subgroups that will not give a connected graph.
Implementation ... not so easy

No problem for ‘small’ part-sizes $m$ and $n$, but ...

The numbers of transitive permutation groups of degrees 24, 32, 36 and 40 are 25000, 2801324, 121279 and 315842, respectively. In particular, this makes the cases in which $m, n \in \{24, 32, 36, 40\}$ quiet challenging, and the case where $(m, n) = (32, 32)$ close to ridiculous.

For other cases with $m = 32$ we simply swapped $m$ and $n$. Even so, the cases $(m, n) = (24, 24)$, $(24, 32)$ and $(24, 36)$ required some special tricks, such as considering interactions of subgroups of $P$ with a normal Hall subgroup of $P$.

Result: All worthy bipartite locally arc-transitive graphs of order up to 63, with blow-ups giving all unworthy ones too.
Case (b): Edge- and vertex-transitive graphs

In this case, we can take the standard approach (using the database of transitive permutation groups of small degree) to construct all vertex-transitive graphs of order up to 47, and then check which ones are edge-transitive, etc.

[Note: The transitive groups of order 48 have not yet been determined, so 47 is a hard upper limit at this stage.]

But we can go further, in the bipartite case ...

• All arc-transitive bipartite graphs of order up to 63 can be found in the same way (and at the same time) as those that are edge- but not vertex-transitive, and

• All half-arc-transitive bipartite graphs of order up to 63 can be found in a similar way, by taking two equal-sized orbits of the stabiliser $P_1$ on the auxiliary set $B$. 
**Computation time:** Several days (using MAGMA)

**Result:** All edge-transitive graphs of order up to 47, and all bipartite edge-transitive graphs of order 48 to 63 as well.

**Summary:**

- **1894 ET graphs of order up to 47**
  - 1429 bipartite, 465 non-bipartite
  - 625 worthy, 1269 unworthy
  - 678 vertex-transitive, 1216 not vertex-transitive

- **3309 bipartite ET graphs of order up to 63**
  - 792 worthy, 2517 unworthy
  - 435 vertex-transitive, 2874 not vertex-transitive.
Questions by Folkman

At the end of his pioneering 1967 paper on semi-symmetric graphs, Folkman asked eight questions. Two of these were general questions about the orders and valencies of semi-symmetric graphs, and they remain open.

Another three were about the existence of semi-symmetric graphs of order $2n$ and valency $d$ where $d$ is prime, or $d$ is coprime to $n$, or $d$ is a prime that does not divide $n$, and these have been answered by the construction of semi-symmetric 3-valent graphs (including examples of orders 110 and 112).

Another question asked if there is a semi-symmetric graph of order 30, and this was answered by Ivanov, who proved in 1987 that no such graph exists.
Another question asked if there is a semi-symmetric graph of order $2pq$ where $p$ and $q$ are odd primes such that $p < q$, and $p$ does not divide $q - 1$. This was answered by Du and Xu, who (in 2000) found all semi-symmetric graphs of order $2pq$ where $p$ and $q$ are distinct primes, and these include graphs of orders $70 = 2 \cdot 5 \cdot 7$, $154 = 2 \cdot 7 \cdot 11$ and $3782 = 2 \cdot 31 \cdot 61$.

The remaining question: Does there exist a semi-symmetric graph of order $2n$ and valency $d$ where $d \geq n/2$?

Our computations answer this positively by producing a few examples, including some with $(2n, d) = (20, 6)$, $(24, 6)$ and $(36, 12)$, giving ratio $d/n = 3/5$, $1/2$ and $2/3$.

Some of these point the way to a construction that proves something even stronger ...
**Construction 1**

For every integer $r \geq 3$, let $A$ be the union of two disjoint sets $A_1$ and $A_2$ of size $r$, and let $B = A_1 \times A_2$, and make these the parts of a bipartite graph $X$ in which an edge joins each $(a_1, a_2) \in B$ to every $a \in A_1 \setminus \{a_1\}$ and every $a \in A_2 \setminus \{a_2\}$.

Then $X$ has $2r + r^2$ vertices and $2r^2(r-1)$ edges: each $a \in A$ has valency $r(r-1)$ and each $b \in B$ has valency $2(r-1)$.

Also the automorphism group of the graph is isomorphic to the wreath product $S_r \wr C_2$, of order $2(r!)^2$, and this acts transitively on the edges of $X$, so $X$ is ET.

Now take the $(r,2)$ blow-up of $X$. The result is an edge-transitive regular graph of order $4r^2$ and valency $2r(r-1)$, but it is not vertex-transitive, since $r > 2$. Hence this graph is semi-symmetric, with ratio $d/n = 2r(r-1)/(2r^2) = (r-1)/r$. 
This gives **an even better answer to Folkman’s question**:

Construction 1 gives an infinite family of semi-symmetric graphs of order $2n = 4r^2$ and valency $d = 2r(r-1)$ in which the ratio $d/n = (r-1)/r$ can be arbitrarily close to 1.

Note: In case you’re wondering, the ratio for the complete bipartite graph $K_{n,n}$ is $n/n = 1$, but this graph is not semi-symmetric: it is arc-transitive.

**Complaint**: The semi-symmetric graphs in Construction 1 (and all other examples of order less than 63) are **unworthy**.

**Question**: Are there any **worthy** examples with large $d/n$?
Construction 2 (using some geometry)

For every odd prime power $q \geq 3$, let $Q$ be the generalised quadrangle associated with a symplectic form on $V = \mathbb{F}_q^4$ (such as $\langle x, y \rangle = x_1y_3 + x_2y_4 - x_3y_1 - x_4y_2$).

Then $Q$ has $q^3 + q^2 + q + 1$ points (1-dimensional subspaces of $V$) and $q^3 + q^2 + q + 1$ isotropic lines (2-dimensional subspaces $U$ of $V$ with $U = U^\perp$), and the associated Levi/incidence graph is locally arc-transitive, but not vertex-transitive (since the geometry is not self-dual, by a theorem of Benson (1970)).

Now take the bipartite complement, in which each point is joined to each of the isotropic lines that do not contain it. This graph is also semi-symmetric, with valency $q^3 + q^2$, and its automorphism group is Aut($PSp(4, q)$), which acts primitively on both parts. Hence the graph is worthy.
This gives a more satisfying answer to Folkman's question:

Construction 2 gives an infinite family of worthy semi-symmetric graphs of order $2(q^3 + q^2 + q + 1)$ and valency $q^3 + q^2$ in which the ratio $d/n = (q^3 + q^2)/(q^3 + q^2 + q + 1)$ can be arbitrarily close to 1.
Edge-transitive maps

A map $M$ is an embedding of a connected graph $X$ into a closed surface $S$, breaking up $S$ into simply connected regions called the faces of $M$.

An automorphism of a map $M$ is a bijection from $M$ to $M$ preserving incidence (between vertices, edges and faces), and then $M$ is called edge-transitive if its automorphism group has a single orbit on edges.
Classification of edge-transitive maps

In a long paper in 1997, Graver and Watkins showed there are 14 different classes of edge-transitive maps. Each class can be defined by what happens ‘locally’ around a given edge \( e \) (and edges incident with \( e \)), or by a universal group.

The principal class ‘1’ consists of regular maps, with group

\[
\mathcal{U} = \langle a, b, c \mid a^2 = b^2 = c^2 = (ac)^2 = 1 \rangle
\]

where \( \langle a, b \rangle, \langle a, c \rangle \) and \( \langle b, c \rangle \) represent the pre-images in \( \mathcal{U} \) of stabilisers of an incident vertex, edge and face, respectively. The universal group for the class ‘2\(^P\)ex’ of chiral maps is isomorphic to the subgroup generated by \( x = ab \) and \( y = bc \), with presentation \( \langle x, y \mid (xy)^2 = 1 \rangle \).

The universal groups of all 14 classes are subgroups \( H \) of index 1, 2 or 4 in \( \mathcal{U} \) with the property that \( \mathcal{U} = H\langle a, c \rangle \).
Questions about edge-transitive maps

The initial work on this topic by Graver and Watkins was taken further by Širáň, Tucker and Watkins (2001), giving examples of ET maps all 14 types — including examples of each type with automorphism groups isomorphic to $S_n$ for all $n \geq 11$ such that $n \equiv 3$ or $11 \mod 12$, etc.

They also posed several questions at the end of their paper — e.g. about finding the smallest genus of surfaces carrying an ET map of given type, finding all examples for genus 2, and determining the genera of surfaces carrying at least one ‘non-degenerate’ ET map $M$ (with no loops or multiple edges in the underlying graph of $M$ or its dual).

Some of these questions were answered by Alen Orbanić in his PhD thesis (Ljubljana, 2006). Others were left open.
Answers to some questions about ET maps

**ET maps with abelian automorphism group**

This question was considered at a recent BIRS workshop on ‘Symmetries of Surfaces, Maps and Dessins’ at Banff (in September 2017), and has now been resolved as follows:

- Orientable maps with $\text{Aut} M$ abelian all have type $4^P$, and those with $\text{Aut}^+ M$ abelian can also have type 2, 2$^*$ or 3.
- Non-o maps with $\text{Aut} M$ abelian have types 3, 4, 4$^*$ or 4$^P$.

**Does some surface carry ET maps of all 14 types?**

Yes! There is one of each type on a surface of genus 14.

[MC (in 2017), although Alen Orbanić got very close to it]
Does some graph underly an ET map of each type?

This question is still open. There is an 8-valent graph of order 16 that underlies an ET map of 11 of the 14 types (namely all except $2^*_\text{ex}$, $2^P\text{ex}$ and 5), and a 6-valent graph of order 36 that underlies an ET map of 10 of the 14 types (namely all except $4^*$, $4^P$, 5 and $5^P$).

Finally ...
An advertisement for the 41st Australasian Conference on Combinatorial Mathematics and Combinatorial Computing the week 10-14 December 2018 in Rotorua, New Zealand
Thank you for taking part in our workshop this week!