Bounded topology of complete manifolds with nonnegative Ricci curvature and quadratically nonnegatively curved infinity

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**Motivation**

**Theorem (Cheeger-Gromoll 1972)**

Let $M^n$ be an $n$-dim complete noncompact Riemannian manifold with $\text{sec} \geq 0$, then $M$ contains a soul $S \subset M$, which is a closed totally convex submanifold, such that $M$ is diffeomorphic to the normal bundle over $S$.

In particular, such a manifold is of finite topological type (or has finite topology), that is to say, it is homeomorphic to the interior of a compact manifold with boundary.

**Question:** Is there any finiteness result for complete Riemannian manifolds with nonnegative Ricci curvature?
• For $n = 2$, YES. All notions of curvature coincide.

• For $n = 3$, YES. Such manifolds were first studied by Schoen and Yau by using stable minimal surfaces and recently classified by G. Liu (2013).

• For $n \geq 4$, NO. Require some additional conditions.
From now on, let $M^n$ be an $n$-dimensional complete noncompact manifold, $p_0 \in M$ a fixed point, denoted by $(M^n, p_0)$.

1. A point $p \in M^n (p \neq p_0)$ is said to be a critical point of the distance $r(p) = d(p_0, p)$ if and only if for all $v \in T_p M^n$ there is a minimal geodesic $\gamma$ from $p$ to $p_0$ such that $\angle(\gamma'(0), v) \leq \frac{\pi}{2}$.

2. **Isotopy Lemma** If $\overline{B(p_0, R_2)} \setminus B(p_0, R_1)$ ($R_1 < R_2 \leq \infty$) contains no critical point of the distance function $r$ to $p_0$, then $\overline{B(p_0, R_2)} \setminus B(p_0, R_1)^{\text{homeo}} \cong \partial B(p_0, R_1) \times [R_1, R_2]$. 
Basic Notions-diameter growth

- extrinsic diameter of geodesic sphere:

\[ \text{diam}(S(p_0, t)) = \sup \{ d_M^n(x, y) : x, y \in S(p_0, t) \}. \]

- interior diameter:

\[ \text{diam}(p_0; t) = \sup \text{diam}(\Sigma_i, M \setminus B(p_0, \frac{1}{2} t)). \]

where \( \Sigma_i \) is a connected component of \( S(p_0, t) \).

- Let \( f : \mathbb{R}^+ \to \mathbb{R}^+ \) be a monotonic function. \((M^n, p_0)\) is said to have a diameter growth of order \( o(f) \) (resp. \( O(f) \)) if and only if \( f^{-1}(t) \text{diam}(p_0; t) \xrightarrow{t \to \infty} 0 \) (resp. remains bounded).
The results of finite topology

**Theorem (Abresch-Gromoll 1990)**

If \((M^n, p_0)\) satisfies

\[
Ric_{M^n} \geq 0, \\
K \geq -c^2, \\
diam(p_0; t) = o(t^{\frac{1}{n}}).
\]

Then \(M^n\) is of finite topological type.

The proof is based on the theory of critical point of distance function \((K \geq -c^2)\) and the estimate of excess function \((Ric_{M^n} \geq 0)\).
The results of finite topology

Set

\[ K_{p_0}(t) = \inf_{M^n \setminus B(p_0, t)} K. \]

**Theorem (Sha-Shen 1997)**

If \((M^n, p_0)\) satisfies

\[ \text{Ric}_{M^n} \geq 0, \]

\[ K_{p_0}(t) \geq -\left( \frac{c}{1 + t^\alpha} \right)^2 \quad (c > 0, \ 0 \leq \alpha \leq 1), \]

\[ \mu = \lim_{t \to \infty} \frac{\text{vol}(B(p_0, t))}{t^n} > 0, \]

\[ \frac{\text{vol}(B(p_0, t))}{t^n} = \mu + o\left( \frac{1}{t^{(n-1)(1-\alpha)}} \right). \]

Then \(M^n\) is of finite topological type.
The results of finite topology

Along this line, we slightly improve the results

**Theorem (J-Yang 2016)**

If \((M^n, p_0)\) satisfies

\[
\begin{align*}
Ric_{M^n} &\geq 0, \\
K_{p_0}(t) &\geq -\left(\frac{c}{1 + t^\alpha}\right)^2 (c > 0, \ 0 \leq \alpha \leq 1), \\
diam(p_0; t) &< \delta(n, c, \alpha) t^{\frac{(n-1)\alpha + 1}{n}} \text{ for } t \text{ large enough.}
\end{align*}
\]

Then \(M^n\) is of finite topological type.
The examples of infinite topology

- (Sha-Yang 1989) There exists a complete 7-dimensional manifold with positive Ricci curvature, sectional curvature bounded below and diameter growth $O(r^{\frac{2}{3}})$, which is of infinite topological type.

- (Menguy 2000) There exists a complete 4-dimensional manifold with positive Ricci curvature, sectional curvature bounded below and maximal volume growth, which is of infinite topological type.

- (Menguy 2000) There exists a complete 4-dimensional manifold with positive Ricci curvature and bounded diameter growth, which is of infinite topological type.
The examples of infinite topology

Based on their construction, I make a summary of their examples:

For $0 \leq \alpha \leq 1$, there exists a complete 4-dimensional manifold $M^4$ with a base point $p_0$ satisfying

$$\text{Ric}_{M^4} > 0,$$

$$K_{p_0}(t) \geq -\left(\frac{h(t)}{1 + t^{2\alpha - 1}}\right)^2,$$

$$\text{diam}(p_0; r) = O(r^\alpha),$$

$$\text{vol}(B(p_0, r)) = O(r^{(n-1)\alpha + 1}),$$

and $M^4$ is of infinite topological type. Where $h(t)$ is a positive increasing function going to infinity when $t$ is going to infinity.
Sha-Shen’s question

**Question:** Is a complete Riemannian manifold \((M^n, p_0)\) with \(\text{Ric} \geq 0\) and \(K_{p_0}(r) \geq -\left(\frac{c}{1+r}\right)^2\) for some \(c > 0\) always of finite topological type?

Note that the condition of sectional curvature here is sometimes called lower quadratic curvature decay (with constant \(c^2\)) or quadratically nonnegatively curved infinity. Similarly, we can define quadratic curvature decay.
Let \((M^n, p_0)\) be a complete Riemannian manifold with \(\text{Ric} \geq 0\) and \(K_{p_0}(r) \geq -\left(\frac{c}{1+r}\right)^2\).

- (Sha-Shen 1997) \(\lim_{t \to \infty} \frac{\text{vol}(B(p_0,t))}{t^n} > 0 \implies \text{finite topology}\)
- (Sha-Shen 1997) \(\lim_{t \to \infty} \frac{\text{vol}(B(p_0,t))}{t} < +\infty \implies \text{finite topology}\)
- (J-Yang 2016) \(\text{diam}(p_0; t) < \delta(n, c)t \implies \text{finite topology}\)

Note that for a complete Riemannian manifold \((M^n, p_0)\) with \(\text{Ric} \geq 0\), we always have

\[\text{diam}(p_0; t) \leq 10^n t.\]
Answer to the conjecture

- For $n = 2$, TRUE.
- For $n = 3$, TRUE.
- For $n = 4, 5$, UNKNOWN.
- For $n \geq 6$, FALSE.
Why do I focus on this conjecture?

- (Abresch 1985) If \((M^n, p_0)\) satisfies that \(\int_0^\infty tK^-_{p_0}(t)dt < +\infty\), here \(K^-_{p_0}(t) = \max\{-K_p(t), 0\}\), then it has finite topology.

- (Abresch 1985) Let \(\lambda : [0, \infty) \to [0, \infty)\) be a continuous function such that \(\int_0^\infty t\lambda(t)dt = +\infty\). Then every noncompact, connected surface \((M^2, p_0)\) carries a complete \(C^2\) metric \(g\) with curvature \(K(p) = -\lambda(d(p_0, p))\) at any point \(p \in M^2\).

- (Gromov 1982 and also Lott-Shen 2000) Any (smooth paracompact) noncompact manifold \(M^n\) admits a complete Riemannian metric with quadratic curvature decay.
The answer to Sha-Shen’s question for $n \geq 6$

**Theorem (J-Yang 2017)**

There exists a complete Riemannian manifold $M^6$ with the base point $p_0$ satisfying

\[
\begin{align*}
Ric &> 0, \\
K_{p_0}(t) &\geq -\left(\frac{K_0}{1 + t}\right)^2, \\
\lim_{t \to \infty} \frac{vol(B_t(p_0))}{t^6} &= 0, \\
diam(p_0; t) &= O(t),
\end{align*}
\]

which is of infinite topological type.
Remarks

- Taking a metric product of the 6-dimensional example with standard spheres, we can get higher dimensional counterexamples which remain these properties.

- Note that our finite result of diameter growth does not reject our example, since \( \delta(n, K_0) = \frac{2K_0 - \cosh^{-1}(\cosh^2 K_0)}{16} \left( \frac{1}{K_0} \right)^{\frac{n-1}{n}} \) goes to 0 when \( K_0 \) goes to infinity.
The construction of Menguy’s example

He starts with $ds^2 = dt^2 + u^2(t)[dx^2 + f^2(t, x)d\sigma^2]$, which almost looks like a 4-dim metric cone

$$ds^2 = dt^2 + (ct)^2[dx^2 + (R_0 \sin x)^2 d\sigma^2]$$

where $0 < c < 1$ and $R_0 \ll 1$, $d\sigma^2$ is the canonical metric of $S^2$.

Choose an increasing sequence \( \{t_i\}_{i=1}^{+\infty} \) with $t_i \xrightarrow{i \to +\infty} +\infty$ and set $r_i = \frac{t_i}{h(t_i)}$, here $h(t)$ is an increasing function slowly going to infinity as $t \to \infty$. For $t_i < t < t_i + 2r_i$, set

$$u(t) = \frac{1}{\sqrt{K_i}} \sin(\sqrt{K_i}(t - t_i + \psi_i)).$$

And

$$u(t) \sim ct.$$
The construction of Menguy’s example

Then

remove the geodesic ball $B^{4}_{\frac{4}{5}r_{i}}(o_{i})(o_{i} = (t_{i} + r_{i}, 0))$
and glue in a $\mathbb{CP}^{2}$ via a neck of Perelman.

During this process, to preserve positive Ricci curvature, we need

(Perelman 1997) Let $M_{1}, M_{2}$ be compact smooth manifolds of positive Ricci curvature, with isometric boundaries $\partial M_{1} \cong \partial M_{2} = X$. Suppose that the normal curvatures of $\partial M_{1}$ are bigger than the negatives of the corresponding normal curvatures of $\partial M_{2}$. Then the result $M_{1} \bigcup_{X} M_{2}$ of gluing $M_{1}$ and $M_{2}$ can be smoothed near $X$ to produce a manifold of positive Ricci curvature.
The construction of Menguy’s example

**Perelman’s Neck:** Let \( (S^n, g = dt^2 + B^2(t)d\sigma^2) \) be a rotationally symmetric metric satisfying

(i) sectional curvature > 1;

(ii) \( 0 \leq t \leq \pi R, \max_t\{B(t)\} = R_0 \), there exists \( \rho \) sufficiently small such that \( 0 < R_0 < \rho < R \) and \( R_0^{n-1} < \rho^n \).

Then there exists a metric of \( S^n \times [0, 1] \) such that

1) \( Ric > 0 \);

2) the boundary component \( S^n \times \{0\} \) is concave, with normal curvatures equal to \( -\lambda \), and is isometric to \( S^n(\rho \lambda^{-1}) \), for some \( \lambda > 0 \);

3) the boundary component \( S^n \times \{1\} \) is strictly convex, with all its normal curvatures bigger than 1, and is isometric to \( (S^n, g) \).
The construction of Menguy’s example

We can continue this surgery when $i$ is going to infinity. Then we get a manifold with infinite second Betti number, which is of course of infinite topological type.

Since $r_i = \frac{t_i}{h(t_i)}$, we have

$$K_{p_0}(t) \geq -\left(\frac{h(t)}{1 + t}\right)^2.$$ 

Since $u(t) = O(t)$, we have

$$\text{vol}(B_t(p_0)) = O(t^4).$$
The construction of our counterexample

Note that $h(t)$ is essential to keep $-\frac{u_{tt}}{u} \geq 0$ so that Ricci curvature can be controlled to be positive, which forces the sectional curvature to be a little weaker than quadratically nonnegatively curved infinity.

In our construction, we set $r_i = rt_i = O(t)$ to get exact quadratically nonnegatively curved infinity. To ensure that Ricci curvature is positive, we add the second product part $\times g(t) S^2$ to control the curvatures through the newly added directions (i.e. $-3 \frac{u_{tt}}{u} - 2 \frac{g_{tt}}{g} \geq 0$), that is a doubly warped product metric

$$ds^2 = dt^2 + u^2(t)[dx^2 + \hat{f}^2(t, x)d\sigma^2] + g^2(t)d\theta^2$$

where $d\sigma^2, d\theta^2$ are both standard metrics of $S^2$. 
The construction of the counterexample

To keep the surgery of gluing $\mathbb{CP}^2$’s still working, we set $g(t)$ to be constant for $t_i + \frac{r_i}{6} < t < t_i + \frac{11r_i}{6}$. And $g(t)$ almost looks like $t^\gamma$. Moreover, to obtain positive Ricci curvature, the order $\gamma$ of $g(t)$ must be sufficiently smaller than 1.

Since $r_i = O(t)$, we have

$$K_{p_0}(t) \geq -\left(\frac{K_0}{1 + t}\right)^2.$$ 

Since $u(t) = O(t)$, we have

$$\text{diam}(p_0; t) = O(t).$$ 

Since $g(t) = O(t^\gamma)$, we have

$$\text{vol}(B_t(p_0)) = O(t^{4+2\gamma}).$$
Thanks for your attention!